# CM09-230323Thu

2023년 3월 23일 목요일 12:59

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Announcements

1. HW#8 will be assigned and due at 10:59am on Tue. Mar. 28th.
   a. See PLMS.
2. Two supplementary handouts are uploaded.
   a. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
      Passwords are …
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1.

Overview (Chapter numbers from the main textbook by Papoulis)

9. General Concepts
   1. RP concept
   2. Stationarity and Independence
   5. Gaussian RPs
   6. Poisson RPs
   7. Complex RPs
   2. Time average and Ergodicity
   3. Correlation functions
   4. Measurement of correlation functions
7. Sequence of Random Variables
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

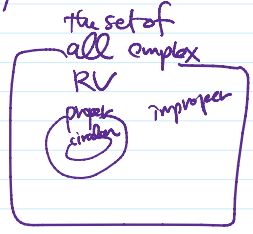
proper → propriety

Review

1. a complex random variable
   a. Def
   b. CDF, PDF
   c. CF?
   d. mean, variance, pseudo-variance
   e. cicularity
   f. propriety vs. impropriety
   g. proper-complex Gaussian random variable
      i. PDF
      ii. CF?

Preview

1. a complex random vector → $\underline{Z}$
   a. Def.
   b. CDF, PDF
   c. mean vector, covariance and pseudo-covariance matrices
   d. propriety vs. impropriety
   e. proper-complex Gaussian random vector
      i. PDF
      ii. CDF of norm
         1) Marcum's Q function
         2) generalized Marcum's Q function
         3) cf) chi-square distribution
2. a complex random process $Z(t)$ . $Z_t$
   a. why?
3. a complex-valued vector random process $\underline{Z}(t)$
4. widely-linear processing of a complex signal. $\underline{Z}[n]$



## Complex-Valued Random Vector

- Def. A complex-valued/complex random vector $\underline{Z}$ of length $n$ consists of two jointly distributed real-valued random vectors $\underline{X}$ and $\underline{Y}$ of length $n$ such that

$$\begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix} = \underline{Z} = \underline{X} + \mathrm{j}\underline{Y} = \begin{bmatrix} X_1 + \mathrm{j}Y_1 \\ \vdots \\ X_n + \mathrm{j}Y_n \end{bmatrix} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{bmatrix} + \mathrm{j}\begin{bmatrix} Y_1 \\ \vdots \\ Y_n \end{bmatrix}$$

- Def. The mean vector, the correlation matrix, the covariance matrix, the pseudo-/complementary correlation matrix, and the pseudo-/complementary covariance matrix of a complex-valued random vector $\underline{Z}$ are defined, respectively, as

$$\begin{aligned} \mu_{\underline{Z}} &= \mathbf{E}[\underline{Z}], \\ R_{\underline{Z}\underline{Z}} &= \mathbf{E}[\underline{Z}\underline{Z}^H] \\ C_{\underline{Z}\underline{Z}} &= \mathbf{E}[(\underline{Z}-\mu_{\underline{Z}})(\underline{Z}-\mu_{\underline{Z}})^H] \\ \tilde{R}_{\underline{Z}\underline{Z}} &= \mathbf{E}[\underline{Z}\underline{Z}^T] \\ \tilde{C}_{\underline{Z}\underline{Z}} &= \mathbf{E}[(\underline{Z}-\mu_{\underline{Z}})(\underline{Z}-\mu_{\underline{Z}})^T]. \end{aligned}$$

- Lemma. A proper-complex Gaussian random vector of length $n$ with mean $\underline{\mu}$ and covariance matrix $\mathbf{C} \succ \mathcal{O}$ has the pdf given by

$$f_{\underline{Z}}(\underline{z}) = \frac{1}{\pi^n \det(\mathbf{C})} \exp\left(-(\underline{z}-\underline{\mu})^H \mathbf{C}^{-1}(\underline{z}-\underline{\mu})\right)$$

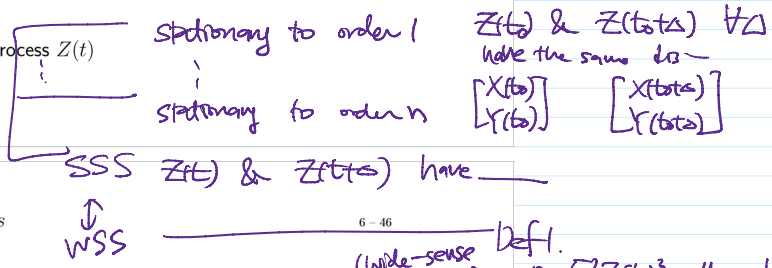
and denoted as $\underline{Z} \sim \mathcal{CN}(\underline{\mu}, \mathbf{C})$



– The real and imaginary parts of the $i$th element of $\underline{z}-\underline{\mu}$ are i.i.d. zero-mean real Gaussian RV with variance $[\mathbf{C}]_{i,i}/2$.

## Complex-Valued Random Process

- (graduate: measure theoretic) Given a probability space $S, \mathcal{A}, P$, a complex-valued random process is a mapping $Z_t(s)$ from the sample space $S$ into a set of complex-valued functions of $t \in \mathbb{I}$, where is an infinite index set.
- (graduate: using two real random processes) Def. A complex-valued/complex random process $Z_t$ consists of two jointly distributed real-valued random processes $X_t$ and $Y_t$ such that
$$Z_t = X_t + \mathrm{j}Y_t,$$
where $t \in \mathbb{I}$ and $\mathbb{I}$ is an infinite index set.
- (textbook) A CT complex-valued random process $Z(t)$ is given by
$$Z(t) = X(t) + \mathrm{j}Y(t)$$
where $X(t)$ and $Y(t)$ are jointly distributed CT real-valued random processes.
- Q. How can we fully characterize a complex-valued random process?
  - A1. by specifying $S, \mathcal{A}, P$ and $Z_t(s)$
  - A2. by specifying all the $n$th-order CDF/PDF/PMF/CFs of $Z(t)$ consistently
  - A3. by specifying all the $n$th-order joint CDF/PDF/PMF/CFs of $X(t)$ and $Y(t)$ consistently
- a stationarity of a complex random process $Z(t)$






  - Def. the stationarity to order $n$
  - Def. $Z(t)$ is called (strict-sense) stationary, if all $n$th-order ... are translation invariant.
    - ∗ Lemma. $Z(t)$ is called stationary, iff $X(t)$ and $Y(t)$ are jointly stationary.

(wide-sense stationary) WSS

Def 1.
(i) $E\{Z(t)\} = \mu_Z, \forall t$.
(ii) $R_{ZZ}(t_1, t_2) \triangleq E\{Z(t_1)\overline{Z(t_2)}^{*}\} = R_{ZZ}(t_1 - t_2), \forall (t_1, t_2)$

- 2nd-order moments of $Z(t)$
  - Def. The mean function/path of $Z(t)$ is defined by
$$\mu_Z(t) = \mathbf{E}[Z(t)] = \mathbf{E}[X(t)] + \mathrm{j}\mathbf{E}[Y(t)]$$
  - Def. The (auto-)correlation function of $Z(t)$ is defined by
$$R_{ZZ}(t+\tau, t) = \mathbf{E}[Z(t+\tau)Z^*(t)]$$
  - Def. The (auto-)covariance function of $Z(t)$ is defined by
$$C_{ZZ}(t+\tau, t) = \mathbf{E}[\{Z(t+\tau) - \mu_Z(t+\tau)\}\{Z(t) - \mu_Z(t)\}^*]$$
  - Def. The complementary correlation function or the pseudo-correlation function of $Z(t)$ is defined by
$$\tilde{R}_{ZZ}(t+\tau, t) = \mathbf{E}[Z(t+\tau)Z(t)]$$
  - Def. The complementary covariance function or the pseudo-covariance function of $Z(t)$ is defined by
$$\tilde{C}_{ZZ}(t+\tau, t) = \mathbf{E}[\{Z(t+\tau) - \mu_Z(t+\tau)\}\{Z(t) - \mu_Z(t)\}]$$

(second-order stationary) SOS

Def 2.
(i) $E\{Z(t)\} = \mu_Z, \forall t$
(ii) $R_{ZZ}(t_1, t_2) = R_{ZZ}(t_1 - t_2), \forall (t_1, t_2)$
(iii) $\tilde{R}_{ZZ}(t_1, t_2) = \tilde{R}_{ZZ}(t_1 - t_2), \forall (t_1, t_2)$

$\Leftrightarrow$ $X(t)$, $Y(t)$ are jointly WSS r.p.

- wide-sense stationarity of $Z(t)$
  1. Def1. $\mu_Z(t) = c, \forall t$ and $C_{ZZ}(t+\tau, t)$ is a function only of $\tau$.
  2. Def2. $\mu_Z(t) = c, \forall t$, and $C_{ZZ}(t+\tau, t)$ and $\tilde{C}_{ZZ}(t+\tau, t)$ are functions only of $\tau$.
  3. Def3. $X(t)$ and $Y(t)$ are jointly WSS real-valued random processes.
  - Def2 = Def3 $\neq$ Def1
  - Some say Def1 as the wide-sense stationarity (WSS), while Def2=Def3 as the second-order stationarity (SOS). I prefer this.
  - Some others say Def2 = Def3, as the wide-sense stationarity (WSS) of a complex random process.
  - Lemma. In either case, if $X(t)$ and $Y(t)$ are jointly WSS, then $Z(t)$ is WSS.

Def. A complex random process is proper if

$$C_{ZZ}(t_1,t_2) \triangleq E\{(Z(t_1)-\mu_Z(t_1))(Z(t_2)-\mu_Z(t_2))\} = 0, \ \forall (t_1,t_2)$$

Otherwise, improper.

# Def. A proper complex Gaussian r.p $Z(t) = X(t) + jY(t)$

① $X(t), Y(t)$ are jointly Gaussian

② proper.

$$\underline{Z}(t) = \begin{bmatrix} Z_1(t) \\ Z_2(t) \\ \vdots \\ Z_n(t) \end{bmatrix}$$



- Def. For two complex processes $Z_i(t)$ and $Z_j(t)$, the cross-correlation and cross-covariance functions are defined by

$$\begin{aligned} R_{Z_iZ_j}(t+\tau, t) &= \mathbf{E}[Z_i(t+\tau)Z_j^*(t)] \\ C_{Z_iZ_j}(t+\tau, t) &= \mathbf{E}[\{Z_i(t+\tau) - \mathbf{E}[Z_i(t+\tau)]\}\{Z_j(t) - \mathbf{E}[Z_j(t)]\}^*]. \end{aligned}$$

- Def. For two complex processes $Z_i(t)$ and $Z_j(t)$, the pseudo-/complementary cross-correlation and pseudo-/complementary cross-covariance functions are defined by

$$\begin{aligned} \tilde{R}_{Z_iZ_j}(t+\tau, t) &= \mathbf{E}[Z_i(t+\tau)Z_j(t)] \\ \tilde{C}_{Z_iZ_j}(t+\tau, t) &= \mathbf{E}[\{Z_i(t+\tau) - \mathbf{E}[Z_i(t+\tau)]\}\{Z_j(t) - \mathbf{E}[Z_j(t)]\}]. \end{aligned}$$

- Def. Two complex processes $Z_i(t)$ and $Z_j(t)$ are jointly WSS if
  - each is a WSS complex-valued random processes and
  - their cross-covariance function is a function of time differences only and not absolute time.
- Def. Two complex processes $Z_i(t)$ and $Z_j(t)$ are uncorrelated processes if $C_{Z_iZ_j}(t+\tau, t) = 0, i \neq j$ and $\tilde{C}_{Z_iZ_j}(t+\tau, t) = 0, i \neq j$.
- Def. Two complex processes $Z_i(t)$ and $Z_j(t)$ are orthogonal processes if $R_{Z_iZ_j}(t+\tau, t) = 0, i \neq j$ and $\tilde{R}_{Z_iZ_j}(t+\tau, t) = 0, i \neq j$.

## Complex-Valued Vector Random Process

- Def. The mean function, the auto-correlation function, the auto-covariance function, the pseudo-/complementary auto-correlation function, and the pseudo-/complementary auto-covariance function of a complex-valued vector random process $\underline{Z}(t)$ are defined, respectively, as

$$\begin{aligned}
\mu_{\underline{Z}}(t) &= \mathbf{E}[\underline{Z}(t)], \\
R_{\underline{Z}\,\underline{Z}}(t+\tau,t) &= \mathbf{E}[\underline{Z}(t+\tau)\underline{Z}^{H}(t)] \\
C_{\underline{Z}\,\underline{Z}}(t+\tau,t) &= \mathbf{E}[\{\underline{Z}(t+\tau)-\mu_{\underline{Z}}(t+\tau)\}\{\underline{Z}(t)-\mu_{\underline{Z}}(t)\}^{H}] \\
\tilde{R}_{\underline{Z}\,\underline{Z}}(t+\tau,t) &= \mathbf{E}[\underline{Z}(t+\tau)\underline{Z}^{T}(t)] \\
\tilde{C}_{\underline{Z}\,\underline{Z}}(t+\tau,t) &= \mathbf{E}[\{\underline{Z}(t+\tau)-\mu_{\underline{Z}}(t+\tau)\}\{\underline{Z}(t)-\mu_{\underline{Z}}(t)\}^{T}].
\end{aligned}$$

- Def. A vector random process $\underline{Z}(t)$ is proper or proper-complex if its pseudo-/complementary covariance function vanishes for all $t$ and $\tau$. Otherwise, improper-complex.
  - Lemma. If a complex Gaussian random process $Z(t)$ is proper, then ... and denoted by $Z(t) \sim \mathcal{CN}(\mu_Z(t), C_{ZZ}(t+\tau,t))$.
  - Lemma. If a complex Gaussian vector-random process $\underline{Z}(t)$ is proper, then ... and denoted by $\underline{Z}(t) \sim \mathcal{CN}(\mu_{\underline{Z}}(t), C_{\underline{Z}\,\underline{Z}}(t+\tau,t))$.

Proper-complex Gaussian Distributions : Part I - Real-valued case

Page 1

○ (Real-valued) Gaussian random vectors

- Def. A random vector $\underline{X}$ with

$$E\{\underline{X}\} = \underline{\mu} \quad \text{and} \quad Cov\{\underline{X}\} = C$$

is called **Gaussian** if its characteristic function is given by

$$\begin{aligned}
\phi_{\underline{X}}(\underline{\omega}) &\triangleq E\{e^{j\underline{\omega}^T\underline{X}}\} \\
&= e^{j\underline{\omega}^T\underline{\mu} - \frac{1}{2}\underline{\omega}^T C\,\underline{\omega}}
\end{aligned}$$

- Remarks

(i) A Gaussian random vector is completely characterized by its **mean** vector $\underline{\mu}$ and the **covariance** matrix $C$.

$\therefore \underline{X} \sim N(\underline{\mu}, C)$ Multivariate Gaussian

(ii) If $C > 0$, then the probability density function is given by

$$f_{\underline{X}}(\underline{x}) = \frac{1}{\sqrt{(2\pi)^N \det C}} \exp\left(-\frac{1}{2}(\underline{x}-\underline{\mu})^T C^{-1}(\underline{x}-\underline{\mu})\right)$$

where $N$ is the dimensionality of $\underline{X}$

(iii) $N=2$, $C = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$, $\rho \neq \pm 1$

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp[$$

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Q. $\underline{Z} \sim CN(\underline{\mu}, C)$

(i) What is the CF of $\underline{Z}$

R -

(ii) simplify it a son as possible

A. Papoulis

$$\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right) + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right\}$$

(i) For $N=2$, $X \sim N(\mu_1, \mu_2; \sigma_1, \sigma_2, \rho)$ or

$X \sim N(\mu_1, \mu_2; \sigma_1^2, \sigma_2^2, \rho)$

↑ bivariate Gaussian

(ii) For $N=1$, $\sigma > 0$,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2}\right)$$

(iii) $X \sim N(\mu, \sigma^2)$ univariate Gaussian

(iii) $X \sim N(\mu, \sigma^2) \Rightarrow E\{(X-\mu)^n\} = \begin{cases} 0, & \text{for odd } n \\ \frac{n!}{(\frac{n}{2})!}\left(\frac{\sigma}{\sqrt{2}}\right)^n & \text{for even } n \end{cases}$

$\mu = 0$

ex) $E\{X^2\} = 1 \cdot \sigma^2$

$E\{X^4\} = 1 \cdot 3 \cdot \sigma^4$

$E\{X^6\} = 1 \cdot 3 \cdot 5 \cdot \sigma^6$

(v) If $C^{-1}$ does not exist, the pdf cannot be written in terms of elementary fts. Dirac delta fts must be used. ← multi-dimensional

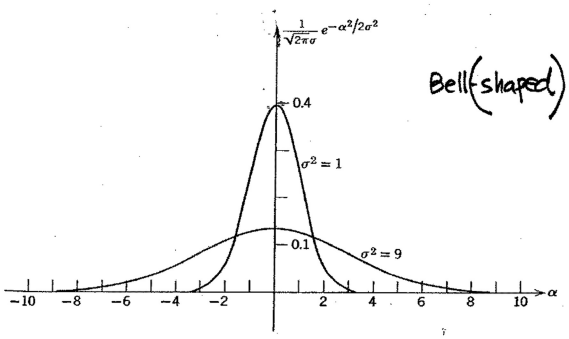
Sometimes, we encounter an impulse fence.

Wozencraft & Jacobs "Principles of Communication Engineering"



variance of a uniformly distributed random variable is $b^2/12$, where $b$ is the width of the density function. Specifying the variance essentially constrains the effective width of the density function. Figure 2.40 illustrates this effect for the Gaussian density function.

A precise statement of the constraint is due to Chebyshev. Let $y$ be a




Bell(shaped) Curve

Figure 2.40 The Gaussian probability density function for two values of variance.

zero-mean random variable with finite variance $\sigma_y^2$. *Chebyshev's inequality* states that for any positive number $\epsilon$

$$P[|y| \geqslant \epsilon] \leqslant \frac{\sigma_y^2}{\epsilon^2}; \qquad \bar{y} = 0. \tag{2.146}$$

Equation 2.146 can be proved as follows. By definition,

$$\overline{y^2} = \int_{-\infty}^{\infty} \alpha^2 p_y(\alpha)\, d\alpha.$$

Since the integrand is positive,

$$\overline{y^2} \geqslant \int_{|\alpha| \geqslant \epsilon} \alpha^2 p_y(\alpha)\, d\alpha.$$

This bound can be weakened further by replacing $\alpha^2$ with its smallest value, $\epsilon^2$, which yields

( s shown in Fig. 2.26, this densit( function may be visualized as tw "fences" of impulses at $\alpha_1 = 1$ and $\alpha_2 = 2$.

For a simple example of the use of a joint density function to calculate a probability, consider the event $A$ defined by

$$A = \{\omega: x_1^2(\omega) + x_2^2(\omega) < c^2\}$$

and the two-dimensional Gaussian density function of Eq. 2.58, with the

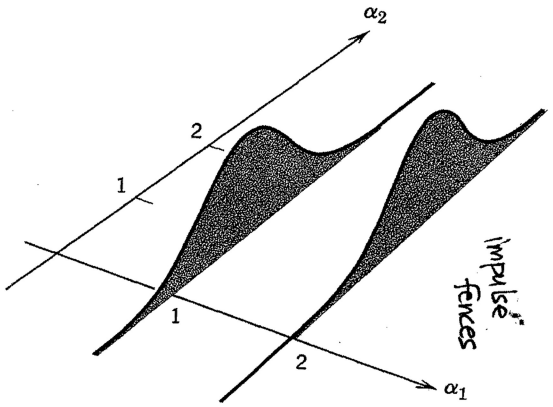



Figure 2.26 Two "fences" of impulses. The value of the one-dimensional impulse at $\alpha_1 = 1$ (or $\alpha_1 = 2$) depends on $\alpha_2$.

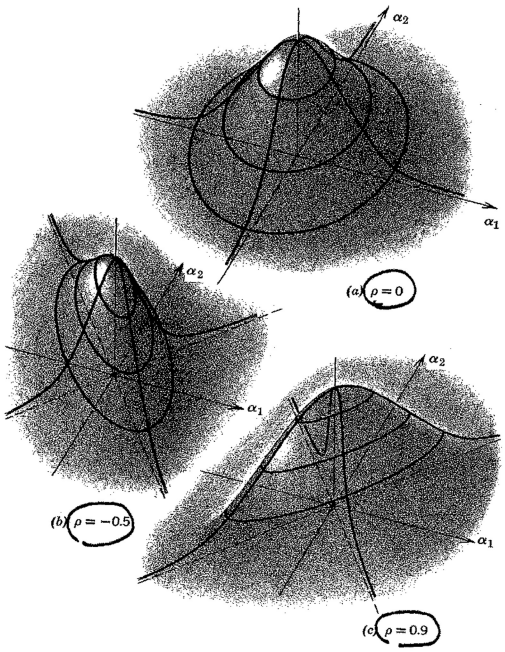



Figure 2.24 Examples of the two-dimensional Gaussian density function.

(v) When $\underline{X} \triangleq \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_N \end{bmatrix}$, any collection of random vectors

$\underline{X}_1, \underline{X}_2, \cdots, \underline{X}_{N'}$,

such as $\underline{X} = \begin{bmatrix} \underline{X}_1 \\ \underline{X}_3 \\ \vdots \\ \underline{X}_{N'} \end{bmatrix}$ are called **Jointly Gaussian.**

- Theorem.

Any transform $\underline{Y}$ of $\underline{X}$ given by

$$\underline{Y} = A\underline{X} + \underline{b} \qquad \underline{X} \in \mathbb{R}^N,\ \underline{b}, \underline{Y} \in \mathbb{R}^M,\ A \in \mathbb{R}^{M \times N}$$

gives a Gaussian random vector $\underline{Y}$

(proof)

$$\phi_{\underline{Y}}(\underline{\omega}) = E\{e^{j\underline{\omega}^T\underline{Y}}\} = E\{e^{j\underline{\omega}^T(A\underline{X}+\underline{b})}\}$$

$$= e^{j\underline{\omega}^T\underline{b}}\, E\{e^{j(A^T\underline{\omega})^T\underline{X}}\}$$

$$= e^{j\underline{\omega}^T\underline{b}}\, e^{j(A^T\underline{\omega})^T\underline{\mu} - \frac{1}{2}(A^T\underline{\omega})^T C (A^T\underline{\omega})}$$

$$= \exp\left(j\underline{\omega}^T(A\underline{\mu}+\underline{b}) - \frac{1}{2}\underline{\omega}^T(ACA^T)\underline{\omega}\right)$$

which is a Gaussian characteristic function with

mean $A\underline{\mu}+\underline{b}$ and covariance $ACA^T$

- Remark

$$C = U\Lambda U^T = \sum_{n=1}^{N} \lambda_n \underline{u}_n \underline{u}_n^T$$

where $U = [\underline{u}_1\ \underline{u}_2 \cdots \underline{u}_N]$ satisfied $U^{-1} = U^T$ and

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \cdots, \lambda_N)$

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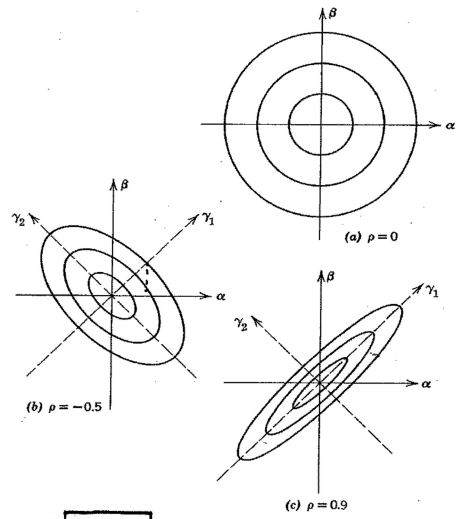



Figure 3.17 Contour plots of constant probability density for the two-dimensional Gaussian density function of Eq. 3.31. The density functions themselves are illustrated in Fig. 2.24 for $\sigma^2 = 1$.

Further insight into the behavior of $p_{n_1,n_2}$ as a function of $\rho$ can be gained from the contour plots of constant probability density shown in Fig. 3.17. The contours are most easily visualized in terms of coordinates $\gamma_1$, $\gamma_2$ rotated 45° from $\alpha$, $\beta$. If we let

$$\alpha = \gamma_1 \cos\frac{\pi}{4} - \gamma_2 \sin\frac{\pi}{4}, \tag{3.35a}$$

$$\beta = \gamma_1 \sin\frac{\pi}{4} + \gamma_2 \cos\frac{\pi}{4}, \tag{3.35b}$$

⇒ Any N-variate Gaussian distribution can be obtained by rotating, reflecting, dilating, translating the Gaussian distribution $N(0, I_N)$

• Remark

Suppose that $X$ and $Z$ are jointly Gaussian r. vectors. Then, the conditional probability distribution of $X$ given $Z=z$ ($X|Z=z$) is also **Gaussian** with mean

$$E\{X|Z=z\} = E\{X\} + Cov(X,Z)Cov(Z)^{-1}(z-E\{Z\})$$

and covariance

$$Cov\{X|Z=z\} = Cov(X) - Cov(X,Z)Cov(Z)^{-1}Cov(Z,X)$$

This result is very important. You can find the usefulness in LMMSE, ALMMSE estimation.

Implications

(i) $E\{X|Z=z\}$ = unconditional mean $E\{X\}$ of $X$ + a linear correction term as a function of $z-E\{Z\}$

⇒ If $z=E\{Z\}$, then $E\{X|Z=z\} = E\{X|Z=E\{Z\}\} = E\{X\}$.

This does not hold in general, $X, Z$ jointly distributed.

(ii) Conditional covariance is less than or equal to un-conditional ".

1-D

# Gaussian distributions for communication engineering

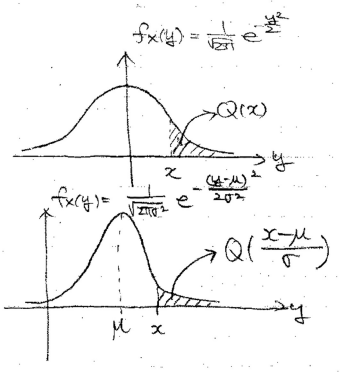
- $X \sim N(0,1)$

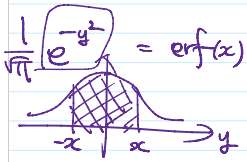
- Definition

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy$$

$X \sim N(\mu, \sigma^2)$

$$P(X \geq \alpha) = Q\left(\frac{\alpha - \mu}{\sigma}\right)$$




$$\frac{1}{\sqrt{\pi}} e^{-y^2} = \operatorname{erf}(x)$$

$$\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$$

- The integral is not an elementary integral. Hence, $F(x) = 1 - Q(x)$ and $\operatorname{erf}(x)$ are distribution function of $X$ (MATLAB built-in) widely tabulated, where

$$\operatorname{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2} dy \quad \text{(error ft)}$$

**erfc(x) = 1 − erf(x)** (complementary error ft)

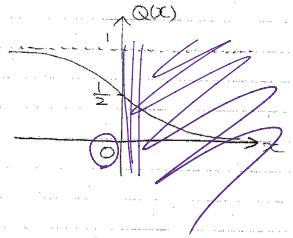
Note

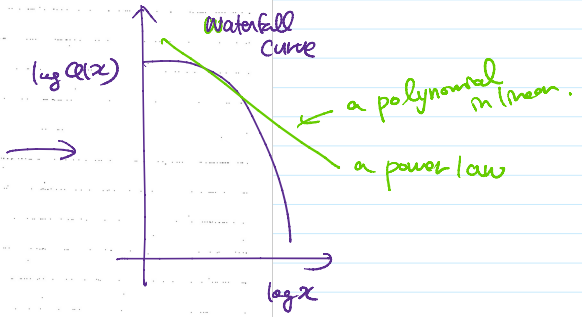
$$Q(x) = \frac{1}{2}\left\{1 - \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)\right\}$$

$$= \frac{1}{2}\operatorname{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

– Properties as a function

(0) (0th order derivative)




$$0 < Q(x) < 1, \quad \forall x \in \mathbb{R}$$

$$Q(x) + Q(-x) = 1, \quad \forall x \in \mathbb{R} \left(\Rightarrow Q(0) = \frac{1}{2}\right)$$

(1) (1st order derivative)

$$Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} < 0 \qquad \forall x \in \mathbb{R}$$

(monotonically decreasing)

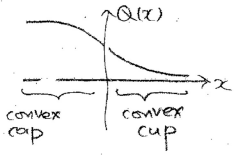


(2) (2nd order derivative)

$$Q''(x) = \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$Q''(x) > 0$ for $x > 0$ (convex cup)

$Q''(x) < 0$ for $x < 0$ (convex cap)




(← Figure. Wozencraft P83. 'Caution' Log scale in y-axis)

$Q(x)$
$Q'(x)$
$Q''(x)$

$Q\left(\frac{1}{\sqrt{x}}\right)$
$Q'\left(\frac{1}{\sqrt{x}}\right)$
$Q''\left(\frac{1}{\sqrt{x}}\right)$