CM10-230328Tue

2023년 3월 23일 목요일 13:03

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Announcements

1. Midterm exam is approaching
2. HW#8 is assigned and due at 10:59am on Thu. Mar. 30th. HW#9 will be assigned soon.
3. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1.

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process

10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. a complex random vector
   a. Def.
   b. CDF, PDF
   c. mean vector, covariance and pseudo-covariance matrices
   d. propriety vs. impropriety
   e. proper-complex Gaussian random vector
      i. PDF

Preview

1. proper-complex Gaussian random vector
   a. CDF of norm
      i. Marcum's Q function
      ii. generalized Marcum's Q function
      iii. cf) chi-square distribution
   b. a complex random process
   c. why?
   d. a complex-valued vector random process
   e. widely-linear processing
2. Time average and ergodicity

I-D Gaussian distributions for communication engineering

- $X \sim N(0,1)$
  - Definition

$$Q(x) \triangleq \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-\frac{y^2}{2}} dy$$

No closed-form expression

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$Y = \frac{X-\mu}{\sigma}$$

$$X \sim N(\mu, \sigma^2)$$

$$F_X(\alpha) \triangleq P(X \geq \alpha) = Q\left(\frac{\alpha-\mu}{\sigma}\right)$$

$Z = 20Y + 60$

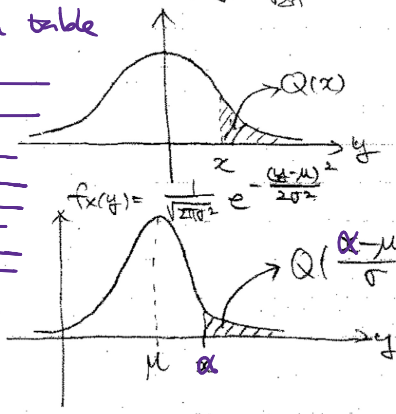
$Z \sim N(60, 20^2)$

Tabulate
= make a table

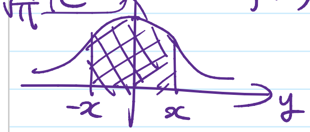
x	$Q(x)$
0	
⋮	
1	
⋮	
2	
⋮	
3	

$f_X(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}$

$f_X(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}$



$\frac{1}{\sqrt{\pi}} e^{-y^2} = \text{erf}(x)$

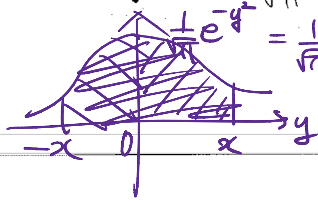


$\text{erfc}(x) = 1 - \text{erf}(x)$

- The integral is not an elementary integral.
  Hence, $F(x) = 1 - Q(x)$ and erf(x) are
  distribution function of X — MATLAB built-in
  widely tabulated, where

$$\textbf{erf}(x) \triangleq \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-y^2} dy \quad \textbf{(error ft)}$$

$$\frac{1}{\sqrt{\pi}} e^{-y^2}, \quad = \frac{1}{\sqrt{\pi}} \int_{-x}^{x} e^{-y^2} dy$$



$$\textbf{erfc}(x) = 1 - \text{erf}(x) \quad \textbf{(complementary error ft)}$$

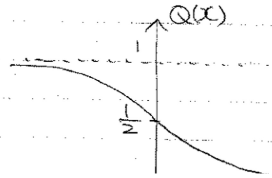
Note

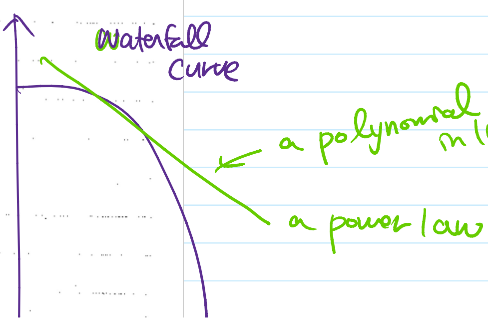
$$Q(x) = \frac{1}{2}\left\{1 - \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right\}$$

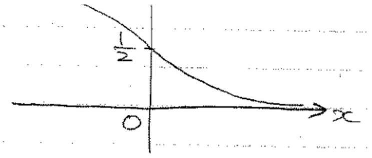
$$= \frac{1}{2}\text{erfc}\left(\frac{x}{\sqrt{2}}\right)$$

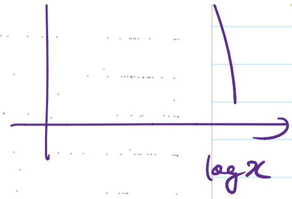
- Properties as a function

(0) (0th order derivative)









a power law

$0 < Q(x) < 1 \ , \ \forall x \in \mathbb{R}$

$Q(x) + Q(-x) = 1 \ , \ \forall x \in \mathbb{R} \ (\Rightarrow Q(0) = \frac{1}{2})$

(i) (1st order derivative)

$$Q'(x) = -\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} < 0 \qquad \forall x \in \mathbb{R}$$

(monotonically decreasing)

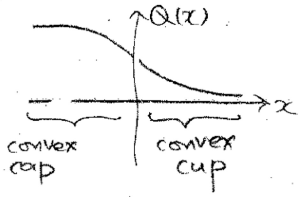
$Q(x)$
$Q'(x)$
$Q''(x)$

$Q\left(\frac{1}{\sqrt{x}}\right)$
$Q'\left(\frac{1}{\sqrt{x}}\right)$
$Q''\left(\frac{1}{\sqrt{x}}\right)$

(ii) (2nd order derivative)

$$Q''(x) = \frac{x}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$Q''(x) > 0$ for $x > 0$ (convex cup)
$Q''(x) < 0$ for $x < 0$ (convex cap)



(← Figure. Wozencraft P83. 'Caution: Log scale in y-axis)

Since

$$0 < \int_\alpha^\infty \frac{1}{\beta^2} e^{-\beta^2/2}\, d\beta < \frac{1}{\alpha^3}\int_\alpha^\infty \beta e^{-\beta^2/2}\, d\beta = \frac{1}{\alpha^3} e^{-\alpha^2/2},$$

we have the bounds

$$\frac{1}{\sqrt{2\pi}\alpha} e^{-\alpha^2/2}\left(1 - \frac{1}{\alpha^2}\right) < Q(\alpha) < \frac{1}{\sqrt{2\pi}\alpha} e^{-\alpha^2/2}; \qquad \alpha > 0. \qquad (2.121)$$

These two bounds are plotted together with $Q(\alpha)$ in Fig. 2.36.

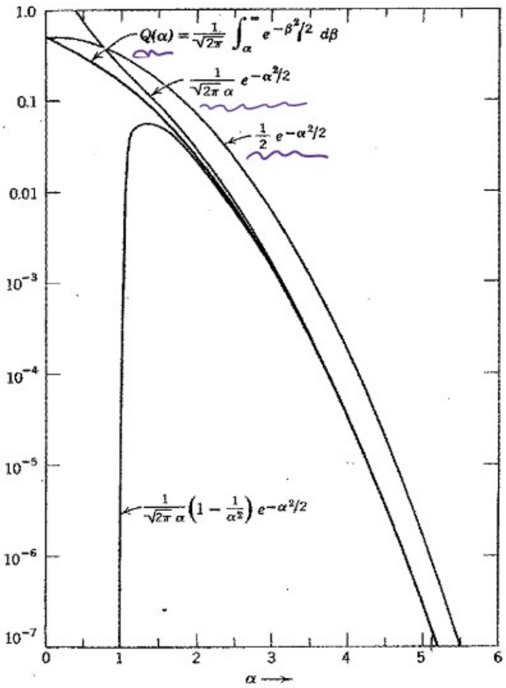


Figure 2.36 The function $Q(\alpha)$ and three bounds.

loose

"an upper bound is tight"

$f(x) \le g(x)$, $\forall x > 0$

– Upper and lower **bounds on Q(x)**

(i) $Q(x) \le e^{-\frac{x^2}{2}}$ $\quad x \ge 0$ (a Chernoff bound)

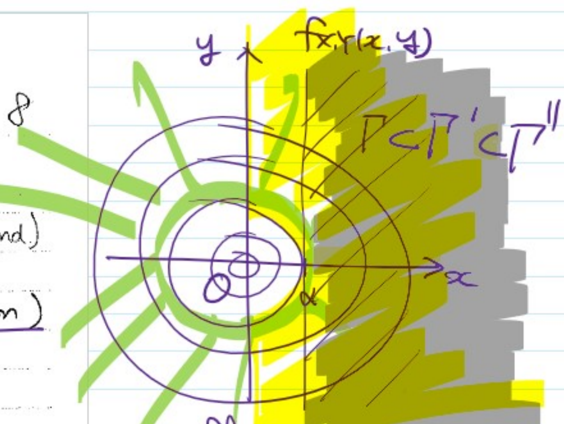
(ii) $Q(x) \le \frac{1}{2} e^{-\frac{x^2}{2}}$ $\quad x \ge 0$ (2-D integration)

In (ii), the equality holds iff $x = 0$

(iii) $\frac{1}{\sqrt{2\pi}}\left(1 - \frac{1}{x^2}\right)\frac{e^{-\frac{x^2}{2}}}{x} < Q(x) < \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{x^2}{2}}}{x}$

for $x > 0$ (Integration by

$X \sim N(0,1)$, $Y \sim N(0,1)$ } indep.



$$\iint_{(x,y)\in\Gamma} f_{X,Y}(x,y)\,dx\,dy = \int_{-\infty}^{\infty}\int_\alpha^\infty f_{X,Y}(x,y)\,dx\,dy$$

$$\frac{1}{\sqrt{2\pi}}\left(1-\frac{1}{x^2}\right)\frac{e^{-\frac{x^2}{2}}}{x} < Q(x) < \frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{x^2}{2}}}{x}$$

for $x > 0$ (Integration by parts)

– Comparison of two upper bounds

$\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{x^2}{2}}}{x}$ is tighter than $\frac{1}{2}e^{-\frac{x^2}{2}}$ for $\sqrt{\frac{2}{\pi}} < x$

$\frac{1}{2}e^{-\frac{x^2}{2}}$ is tighter than $\frac{1}{\sqrt{2\pi}}\frac{e^{-\frac{x^2}{2}}}{x}$ for $0 \le x < \sqrt{\frac{2}{\pi}}$ $(\simeq 0.7979)$ (← See Wozencraft Figure 2.36)

However, actually, both are inaccurate.

$\boxed{\text{– Approximation for } Q(x)}$

(i) erfc(x) can be represented by a Taylor series in terms of the **Hermite polynomial**

$$= \int_{-\infty}^{\infty}\int_{\alpha}^{\infty} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty} f_Y(y)\,dy \int_{\alpha}^{\infty} f_X(x)\,dx$$

$$= 1 \cdot Q(\alpha)$$

$$Q(x) \le \iint_{D'} f_{X,Y}(x,y)\,dx\,dy$$

$$= \frac{1}{2}\iint_{D''} f_{X,Y}(x,y)\,dx\,dy =$$

$$= \frac{1}{2}e^{-\frac{x^2}{2}}, \forall x \ge 0$$

of degree $(K-1)$ as

$$\frac{d^k}{dx^k}\,\mathrm{erfc}(x) = (-1)^k \frac{2}{\sqrt{\pi}}\, H_{k-1}(x)\, e^{-x^2}$$

(ii) Alternative expression for $Q(x)$ is given by

$$Q(x) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{x^2}{2\sin^2\theta}\right) d\theta \quad (x\ge 0)$$

proper ~~definite~~ integral

improper ~~indefinite~~ integral

$$\left(= \frac{1}{\sqrt{2\pi}}\int_x^{\infty} \exp\left(-\frac{y^2}{2}\right) dy \right)$$

Numerical integration! required

$$E[Q(X)]$$
$$= E\left[\int_X^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy\right]$$
$$= E\left[\frac{1}{\pi}\int_0^{\frac{\pi}{2}} \exp\left(-\frac{X^2}{2\sin^2\theta}\right) d\theta\right]$$

The exponential family

(iii) n-th order approximation of $Q(x)$

$$Q(x) \simeq Q_n(x) = \frac{1}{\sqrt{2\pi}\,x} e\left(-\frac{x^2}{2}\right) \times \left\{1 + \sum_{k=1}^{n} (-1)^k \frac{1\cdot 3\cdot 5\cdots(2k-1)}{x^{2k}}\right\}$$

for $x > 0$

- Property of $Q(\sqrt{2x})$ for $x>0$.

(i) $Q'(\sqrt{2x}) < 0$ for $x>0$ monotonically decreasing

(ii) $Q''(\sqrt{2x}) > 0$ for $x>0$ convex cup ft

We can use Jensen's ineq.

(sol)

$$Q'(\sqrt{2x}) = -\frac{1}{\sqrt{2\pi}} e^{-x} \frac{1}{\sqrt{2x}}$$

$$Q''(\sqrt{2x}) = \frac{1}{\sqrt{2\pi}} e^{-x} \frac{1}{\sqrt{2x}} \left(1+\frac{1}{2x}\right)$$

1-D

2-D

- $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

– When $\mu_X = \mu_Y = 0$

(i) Cartesian coordinate

$$f_{XY}(x, y) = \frac{1}{\sqrt{2\pi}^2 \sqrt{\det(\sigma^2 I)}} \exp\left(-\frac{1}{2} [x \ y] (\sigma^2 I)^{-1} \begin{bmatrix} x \\ y \end{bmatrix}\right)$$

$$= \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{x^2}{2\sigma^2}\right)} \ \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(-\frac{y^2}{2\sigma^2}\right)}$$

$$= \quad f_X(x) \quad \times \quad f_Y(y)$$

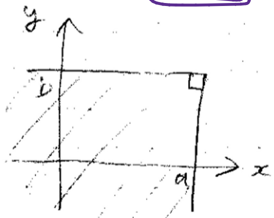
$\boxed{V}$

$\Rightarrow$ X & Y are independent

/ex/ $\Pr(X \le a, Y \le b)$

$= \Pr(X \le a) \Pr(Y \le b)$

$= \{1 - Q(\frac{a}{\sigma})\}\{1 - Q(\frac{b}{\sigma})\}$



(ii) Polar coordinate

If we rotate by

$$\begin{bmatrix} \tilde{X} \\ \tilde{Y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

then

$$\begin{aligned} f_{\tilde{X}\tilde{Y}}(\tilde{x},\tilde{y}) &= f_{XY}(\tilde{x}\cos\theta+\tilde{y}\sin\theta,\ -\tilde{x}\sin\theta+\tilde{y}\cos\theta) \\ &= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{1}{2\sigma^2}\{(\tilde{x}\cos\theta+\tilde{y}\sin\theta)^2 + (-\tilde{x}\sin\theta+\tilde{y}\cos\theta)^2\}\right) \\ &= \frac{1}{2\pi\sigma^2}\exp\left(-\frac{\tilde{x}^2+\tilde{y}^2}{2\sigma^2}\right) \\ &= f_{XY}(\tilde{x},\tilde{y}) \qquad \forall\theta\in[0,2\pi) \end{aligned}$$

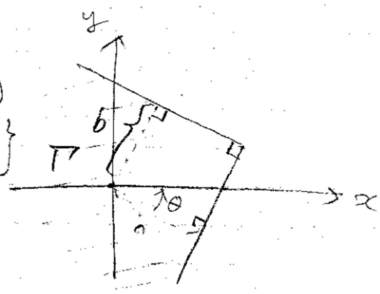
$\Rightarrow$ rotationally invariant distribution

$\equiv$ circularly symmetric distribution

/ex/ $\Pr((X,Y)\in\Gamma)$

$= \Pr(\tilde{X}\le a,\ \tilde{Y}\le b)$

$= \{1-Q(\frac{a}{\sigma})\}\{1-Q(\frac{b}{\sigma})\}$



In the polar coordinate,

$$\begin{aligned} f_{R,\Theta}(r,\theta) &\triangleq f_{XY}(r\cos\theta, r\sin\theta)\, r \\ &= \frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right), \quad r\ge 0,\ 0\le\theta<2\pi \end{aligned}$$

where $R = \sqrt{X^2+Y^2}$, $\Theta = \tan^{-1}\frac{Y}{X}$

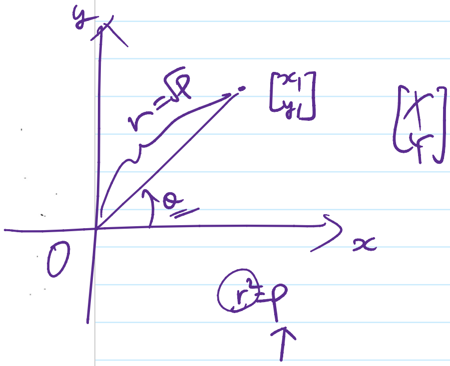
$$= \underbrace{\frac{1}{2\pi}}\cdot\underbrace{\frac{r}{\sigma^2}\exp\left(-\frac{r^2}{2\sigma^2}\right)}$$

$$= f_\Theta(\theta) \times f_R(r)$$

$\Rightarrow$ Θ and R are independent.

Θ: uniform random variable on $[0,2\pi)$

R: Rayleigh random variable w/ parameter σ^2

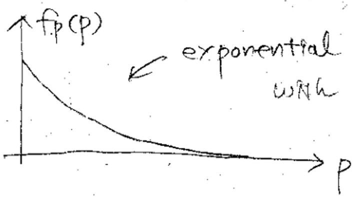


As we've seen $f_{R,\Theta}(r,\theta)$ is not a function of θ. (circularly symmetric!)

Let $P \triangleq R^2$, then

$$f_P(p) = \frac{1}{2\sqrt{p}} f_R(\sqrt{p})$$

$$= \frac{1}{2\sigma^2}\exp\left(-\frac{p}{2\sigma^2}\right), \quad p \ge 0$$



exponential random variable with parameter $\frac{1}{2\sigma^2}$

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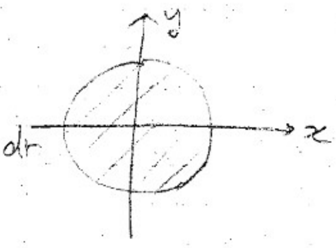
/ex/ $\Pr(\sqrt{X^2+Y^2} \le c)$

$= \Pr(R \le c)$

$= \int_0^c \frac{r}{\sigma^2} \exp(-\frac{r^2}{2\sigma^2})\, dr$



$= \left[-\exp(-\frac{r^2}{2\sigma^2})\right]_0^c$

$= 1 - \exp(-\frac{c^2}{2\sigma^2})$.

Now, let's show

$$Q(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp(-\frac{x^2}{2\sin^2\theta})\, d\theta, \quad x \ge 0$$

and

$$\boxed{Q(x) \le \frac{1}{2}\exp(-\frac{x^2}{2}), \quad x \ge 0}$$

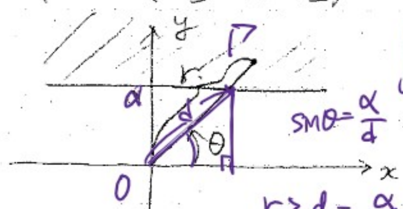
Note that

$$= 2\int_0^{\frac{\pi}{2}}\int_{\frac{\alpha}{\sin\theta}}^{\infty} \frac{1}{2\pi}\frac{r}{1} e^{-\frac{r^2}{2}}\, dr\, d\theta.$$

$$Q(a) = \Pr(Y \ge a, -\infty < X < \infty)$$

where $\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

Hence,



$\sin\theta = \frac{\alpha}{d}$

$r \ge d = \frac{\alpha}{\sin\theta}$

$$\iint_\Gamma f_{XY}(x,y)\, dx\, dy = Q(\alpha) = \int_0^{\pi}\int_{\frac{\alpha}{\sin\theta}}^{\infty} f_{XY}(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

$\Gamma = \{(x,y) : x \in \mathbb{R}, y \ge \alpha\}$

$= \{(r,\theta) : 0 < \theta < \pi, r \ge \frac{\alpha}{\sin\theta}\}$

$$Q(a) = \Pr(R\sin\Theta \ge a, 0 < \Theta < \pi)$$

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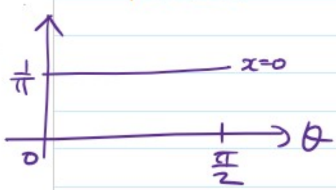
$$= \int_0^{\pi}\int_{\frac{a}{\sin\theta}}^{\infty} \frac{r}{2\pi} \exp(-\frac{r^2}{2})\, dr\, d\theta$$

$$= \int_0^{\pi} \frac{1}{2\pi} \exp(-\frac{a^2}{2\sin^2\theta})\, d\theta$$

$$\boxed{Q(\alpha) = \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp(-\frac{a^2}{2\sin^2\theta})\, d\theta}$$

$$Q(x) = \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp(-\frac{x^2}{2\sin^2\theta})\, d\theta$$



Since $\sin^2\theta \le 1$

$\Rightarrow \frac{1}{\sin^2\theta} \ge 1$

$\Rightarrow -\frac{x^2}{2\sin^2\theta} \le -\frac{x^2}{2}$

$\Rightarrow \exp(-\frac{x^2}{2\sin^2\theta}) \le \exp(-\frac{x^2}{2})$

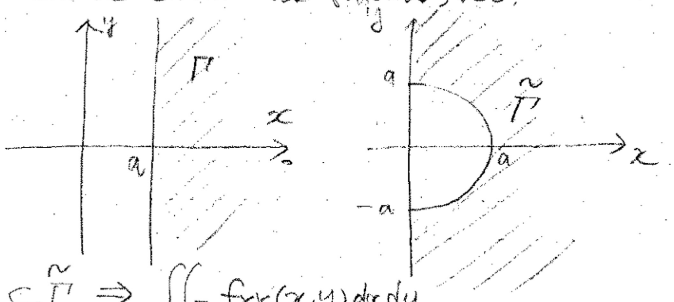
$\Rightarrow Q(x) \le \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp(-\frac{x^2}{2}) \quad x \ge 0$

$$\Rightarrow \exp\left(-\frac{x^2}{2\sin^2\theta}\right) \le \exp\left(-\frac{x^2}{2}\right)$$

$$\Rightarrow Q(x) \le \int_0^{\frac{\pi}{2}} \frac{1}{\pi} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

$$\Rightarrow Q(x) \le \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

This can be shown as follows, too.



$$\Gamma \subset \tilde{\Gamma} \Rightarrow \iint_{\Gamma} f_{XY}(x,y)\,dx\,dy \le \iint_{\tilde{\Gamma}} f_{XY}(x,y)\,dx\,dy$$

$$\Rightarrow Q(x) \le \Pr\left(R \ge x, -\frac{\pi}{2} \le \Theta \le \frac{\pi}{2}\right)$$

$$= \frac{1}{2} \Pr(R \ge x)$$

$$= \frac{1}{2} \exp\left(-\frac{x^2}{2}\right), \quad x \ge 0$$

– When $(\mu_X, \mu_Y) \neq (0, 0)$

(i) Cartesian coordinate

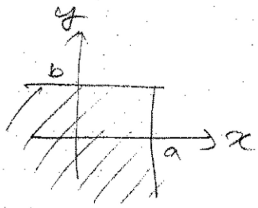
$$f_{XY}(x,y) = \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(x-\mu_X)^2}{2\sigma^2}\right)}_{f_X(x)} \times \underbrace{\frac{1}{\sqrt{2\pi}\,\sigma}\exp\left(-\frac{(y-\mu_Y)^2}{2\sigma^2}\right)}_{f_Y(y)}$$

$\Rightarrow$ X & Y are independent

/ex/ $\Pr(X \le a, Y \le b)$

$= \Pr(X \le a)\Pr(Y \le b)$

$= \left\{1 - Q\left(\frac{a-\mu_X}{\sigma}\right)\right\}\left\{1 - Q\left(\frac{b-\mu_Y}{\sigma}\right)\right\}$



(ii) Polar coordinate

If we rotate with respect to $\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}$ by θ

$$\begin{bmatrix}\tilde{X} \\ \tilde{Y}\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}X-\mu_X \\ Y-\mu_Y\end{bmatrix} + \begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}$$

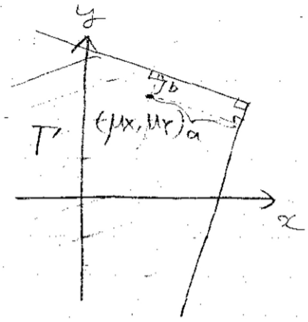
then

$$f_{\tilde{X}\tilde{Y}}(\tilde{x},\tilde{y}) = f_{XY}\big((\tilde{x}-\mu_X)\cos\theta + (\tilde{y}-\mu_Y)\sin\theta + \mu_X,\ -(\tilde{x}-\mu_X)\sin\theta + (\tilde{y}-\mu_Y)\cos\theta + \mu_Y\big)$$

$$\cdots = f_{XY}(\tilde{x},\tilde{y})$$

$\Rightarrow$ circularly symmetric with respect to $\begin{bmatrix}\mu_X \\ \mu_Y\end{bmatrix}$

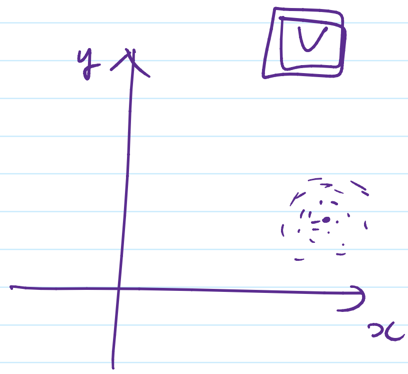
(ex) $\Pr((X,Y)\in\Gamma)$
$$=\left\{1-Q\left(\frac{a}{\sigma}\right)\right\}\left\{1-Q\left(\frac{b}{\sigma}\right)\right\}$$



In the polar coordinate,

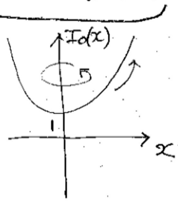
$$f_{R,\Theta}(r,\theta)\triangleq f_{XY}(r\cos\theta, r\sin\theta)\,r$$
$$=\frac{r}{2\pi\sigma^2}\exp\left(-\frac{(r\cos\theta-\mu_X)^2+(r\sin\theta-\mu_Y)^2}{2\sigma^2}\right)$$
$$=\frac{r}{2\pi\sigma^2}\exp\left(-\frac{r^2+\chi^2}{2\sigma^2}\right)\exp\left(\frac{r(\mu_X\cos\theta+\mu_Y\sin\theta)}{\sigma^2}\right)$$

where $\chi\triangleq\sqrt{\mu_X^2+\mu_Y^2}$



Since
$$f_R(r)=\int_0^{2\pi}f_{R\Theta}(r,\theta)\,d\theta$$
$$=\cdots=\frac{r}{\sigma^2}\exp\left(-\frac{r^2+\chi^2}{2\sigma^2}\right)$$
$$\times\underbrace{\frac{1}{2\pi}\int_0^{2\pi}\exp\left(\frac{r\chi}{\sigma^2}\cos(\theta+\alpha)\right)d\theta}_{\triangleq I_0\left(\frac{r\chi}{\sigma^2}\right)}$$

where $I_0(x)\triangleq\frac{1}{2\pi}\int_0^{2\pi}\exp(x\cos\theta)\,d\theta$



is called **the zero-th order modified Bessel function of the 1st kind**.

The distribution of R is a $\left\{ \begin{array}{l} \textbf{Ricean} \\ \textbf{Rician} \end{array} \right\}$ **distribution**

$$f_\Theta(\theta) = \frac{1}{2\pi} \exp\left(-\frac{\alpha^2}{2\sigma^2}\right) + \frac{\alpha \cos(\theta - \theta_0)}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{-\alpha^2 \sin^2(\theta-\theta_0)}{2\sigma^2}\right) \times \left(1 - Q\left(\frac{A_0 \cos(\theta-\theta_0)}{\sigma}\right)\right)$$

where $\alpha = \sqrt{\mu_x^2 + \mu_y^2}$

$\theta = \tan^{-1}\left(\frac{\mu_y}{\mu_x}\right)$

$$\lim_{\frac{\alpha}{\sigma} \to \infty} f_\Theta(\theta) = \delta(\theta - \theta_0)$$

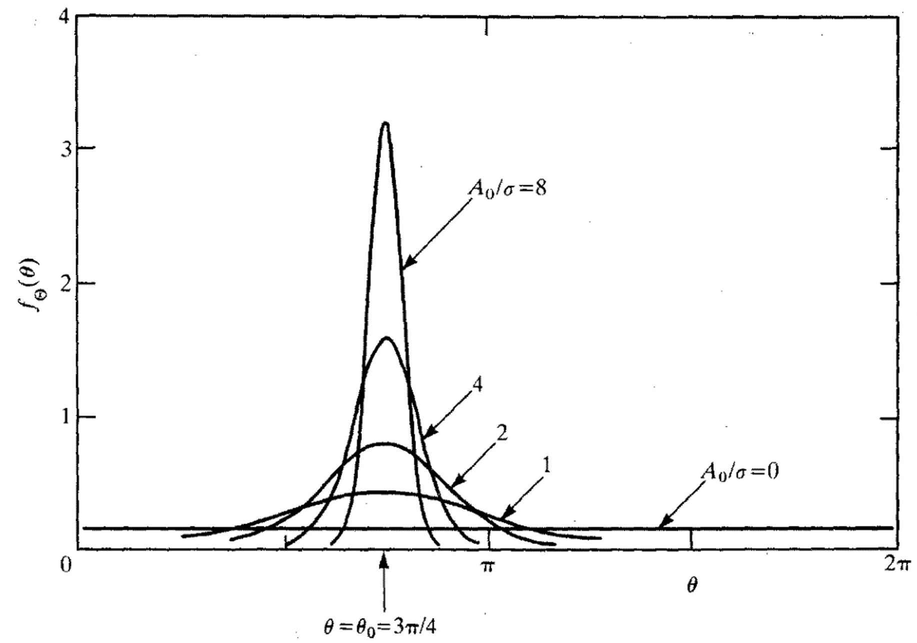


FIGURE 10.6-2
Probability density function of the phase of the sum of a sinusoidal signal and gaussian noise. Curves are plotted for a signal phase of $\theta_0 = 3\pi/4$.

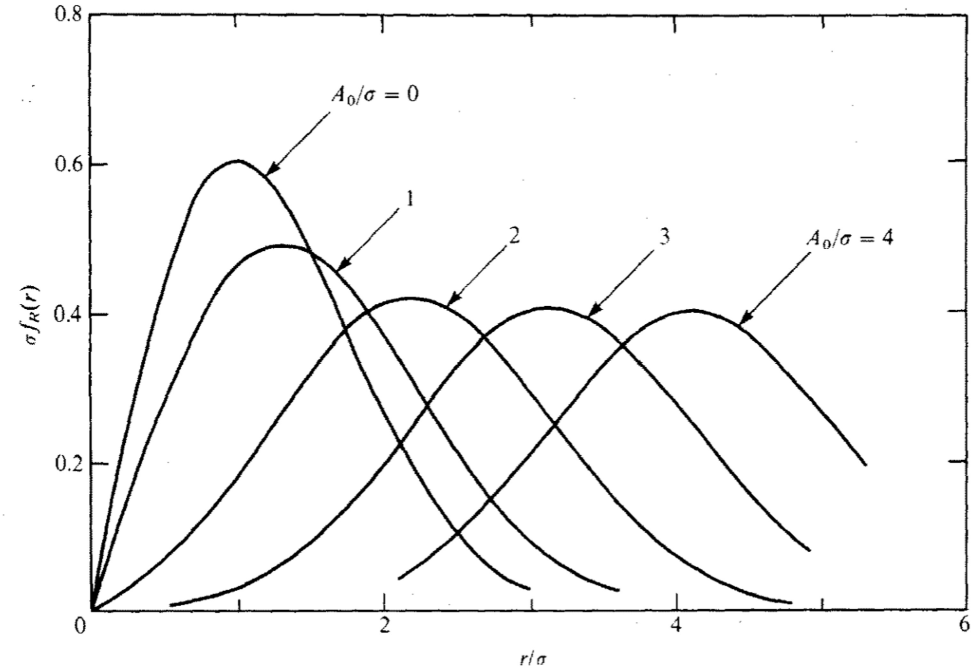


FIGURE 10.6-1
Probability densities of the envelope of a sinusoidal signal (amplitude A_0) plus noise (power σ^2) for various ratios A_0/σ.

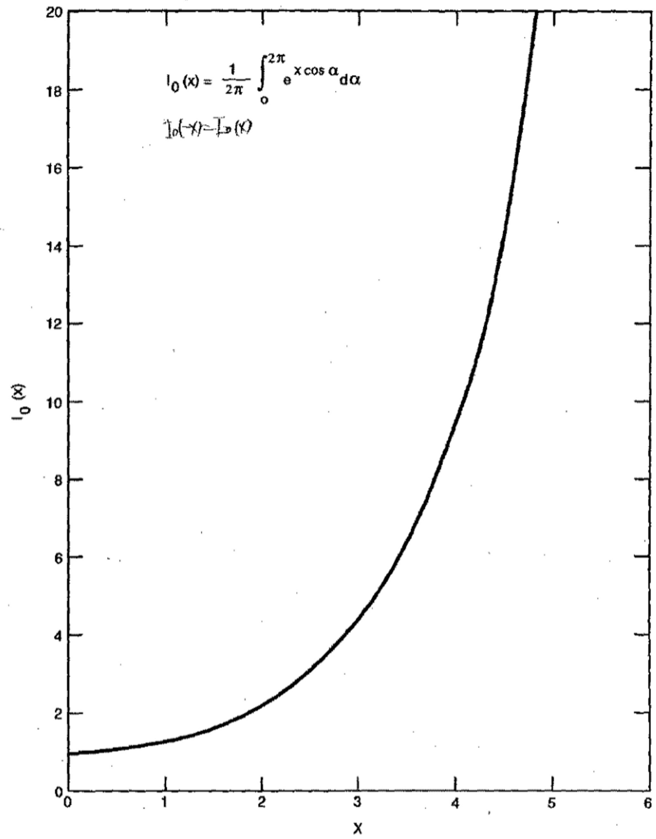


Figure 5.2 Plot of $I_0(x)$

Hence, using (5.22) in (5.20), the optimum decision rule sets $\widehat{m}(\rho(t)) = m_k$ when

$$I_0\left(\frac{2\xi_i}{N_0}\right)\exp\left(-\frac{E_i}{N_0}\right) \tag{5.23}$$

is maximum for $k = i$. Recall from (5.17) that the envelope ξ_i is given by

$$\xi_i = \sqrt{z_{ci}^2 + z_{si}^2} \tag{5.24}$$