2023년 3월 30일 목요일 16:29

CM11-230330Thu

Announcements

1. Midterm exam is approaching.
2. HW#9 will be assigned soon.
3. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1.

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process

10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. real Gaussian random variable
   a. Q function
   b. ~~upper bounds~~
   c. alternative expression of Q function

$Q(\square)$ $\quad$ $\frac{x-\mu}{\sigma}$ $\quad$ [m] $\quad$ $X \quad \mu \quad \sigma$

2. real Gaussian random vector: 2-D
   a. Q function
   b. indep Rayleigh and Uniform vs. dependent Rician and ...

$\begin{bmatrix} X \\ Y \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right)$

(i) $\begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad f_{R\Theta}(r,\theta) = f_{XY}(r\cos\theta, r\sin\theta)\, r = f_R(r) f_\Theta(\theta)$

a Rayleigh RV. $\quad$ uniform $[0, 2\pi)$ $(0, 2\pi]$ $[0, 2\pi]$

$P = R^2$

exponential

(ii) $\begin{bmatrix} \ \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Preview

1. 2-D
   a. Marcum's Q function
2. 2m-D
   a. generalized Marcum's Q function
3. n-D
   a. chi-square distribution for DOF n
4. proper-complex Gaussian random vector
   a. ~~CDF of norm~~
   b. a complex random process
   c. why?
   d. a complex-valued vector random process
   e. widely-linear processing
5. Time average and ergodicity

$Z = \begin{bmatrix} Z_1 \\ Z_2 \\ \vdots \\ Z_n \end{bmatrix}$ $\quad Z_i = \mathrm{Re}\{Z_i\} + j\,\mathrm{Im}\{Z_j\}$

the set of all real RV

exponential family

## ○ 2-D case (cont.)

Given

Def. When $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \sigma^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$, find $\Pr(\underbrace{X_1^2 + X_2^2 \geq \alpha}_{Y})$ tail prob.

$Q\left(\frac{x-\mu}{\sigma}\right)$

$$Q\left(\frac{\chi}{\sigma}, \frac{x}{\sigma}\right) \triangleq \Pr\left(\sqrt{X_1^2 + X_2^2} \geq x\right)$$

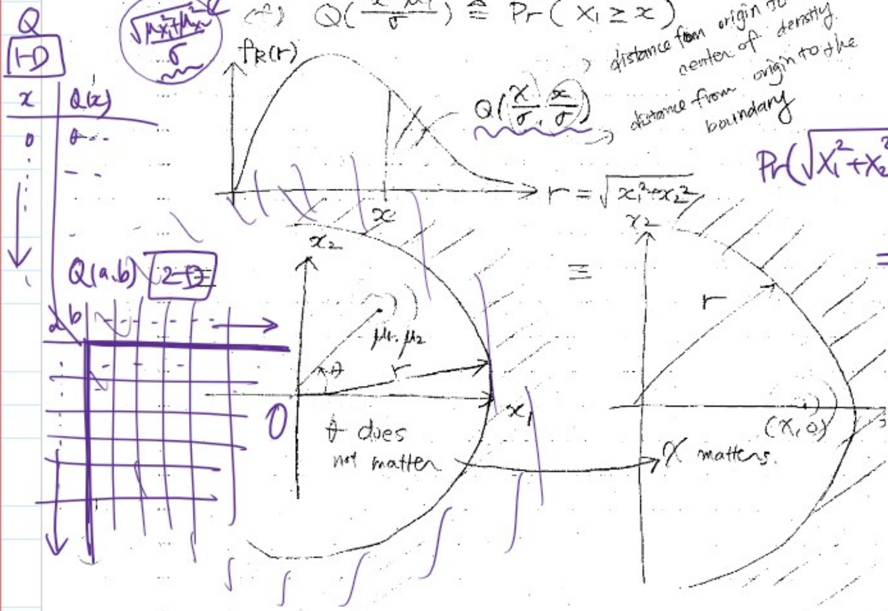
where $\chi \triangleq \sqrt{\mu_1^2 + \mu_2^2}$.

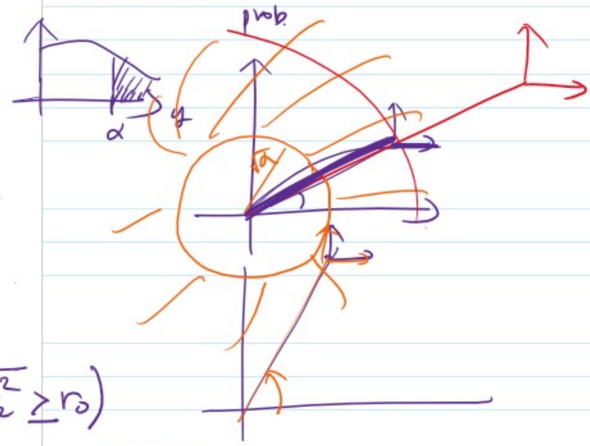
chi /kai/

$\boxed{Q(a, b)}$ is called **Marcum's Q-function**

$a = \frac{\sqrt{\mu_{x_1}^2 + \mu_{x_2}^2}}{\sigma}$

cf) $Q\left(\frac{x-\mu_1}{\sigma}\right) \triangleq \Pr(X_1 \geq x)$




$$\Pr\left(\sqrt{X_1^2 + X_2^2} \geq r_0\right) = Q\left(\frac{\sqrt{\mu_{x_1}^2 + \mu_{x_2}^2}}{\sigma}, \frac{r_0}{\sigma}\right)$$

1-D

$$\Pr(X_1 \geq r_0) = Q\left(\frac{r_0 - \mu_{x_1}}{\sigma_{x_1}}\right)$$

2-D

$$\Pr\left(\sqrt{X_1^2 + X_2^2} \geq r_0\right) = Q\left(\frac{\sqrt{\mu_{x_1}^2 + \mu_{x_2}^2}}{\sigma}, \frac{r_0}{\sigma}\right)$$

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Q 1

Q( ) 2

~~Q(a,b,c) 3-D~~

~~Q(a,b,c,d) 4-D~~

## ○ 2m-D case

The **generalized Marcum's Q-function**

If $Y = \sqrt{\sum_{i=1}^{2m} X_i^2}$, then

$$F_Y(y) = 1 - Q_m\left(\frac{\chi}{\sigma}, \frac{y}{\sigma}\right)$$

$Y = \sqrt{X_1^2 + X_2^2}$

$Y = \sqrt{X_1^2 + X_2^2 + X_3^2 + X_4^2}$

$\underline{Z} = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$

$\underline{Z} \sim CN(\mu_{\underline{Z}}, \sigma^2 I)$

$\underline{Z} - \mu_{\underline{Z}}$

where

$$\boxed{Q_m(a, b)} = Q_1(a, b) + e^{-\frac{a^2+b^2}{2}} \sum_{k=1}^{m-1} \left(\frac{b}{a}\right)^k I_k(ab)$$

w/

$$Q_1(a, b) = e^{-\frac{a^2+b^2}{2}} \sum_{k=0}^{\infty} \left(\frac{a}{b}\right)^k I_k(ab), \quad (b > a > 0)$$

$= Q(a, b)$

$a = \frac{\sqrt{(\mu_{x_1}^2 + \mu_{x_2}^2) + \cdots (\ )^2}}{\sigma}$

unitless

$b = \frac{\ }{\sigma}$

· $Y = \sum_{i=1}^{n} X_i^2$ where $X_i$'s are independent w/ $\sigma^2$.

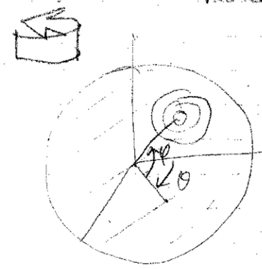
$\equiv$

Given $f_X(x)$, find $f_Y(y)$ w/ $Y = X^2$

∵ $Y = \sqrt{\sum_{i=1}^{m} X_i^2}$ learned already! for some m.

... named!

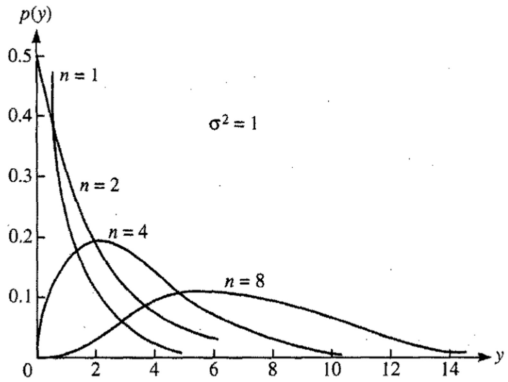
named ;



$\theta \cdot \varphi$ does not matter only the distance to the center (mean) matters.



n=2 exponential ;




central

The pdf of a chi-square-distributed random variable for several degrees of freedom.

*gamma) pdf with n degrees of freedom.* It is illustrated in Fig. 2-1-9. The case $n = 2$ yields the exponential distribution.

The first two moments of $Y$ are

$$E(Y) = n\sigma^2$$

○ n-D case

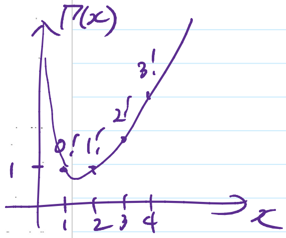
$Y=X^2$, $Y=X_1^2+X_2^2$ 2) $Y=X_1^2+X_2^2+X_3^2$

• $Y=\sum_{i=1}^{n} X_i^2$ where $X_i$'s are i.i.d. $N(0,\sigma^2)$

$Pr(Y \geq y_0)$ — central chi-square / non " chi-square.

The **central chi-square distribution**

$$P_Y(y)=\frac{1}{\sqrt{2\sigma^2}^{\,n}\,\Gamma(\frac{n}{2})}\, y^{\frac{n}{2}-1} e^{-\frac{y}{2\sigma^2}}, \quad y \geq 0$$




a **chi-square** (or **gamma**) **pdf with n degrees of freedom.**

In particular, $n=2$ yield an exponential density. (← Figure. Proakis 3rd. P43)

• $Y=\sum_{i=1}^{n} X_i^2$ where $X_i$'s are independent and $X_i \sim N(\mu_i,\sigma^2)$

~~$Y=\sqrt{X_1^2+X_2^2+\cdots}$~~

$Y=X_1^2+X_2^2+\cdots$

Let

$s=\sqrt{\mu_1^2+\mu_2^2+\cdots+\mu_n^2}$, then

$$P_Y(y)=\frac{1}{2\sigma^2}\left(\frac{y}{s^2}\right)^{\frac{n-2}{4}} e^{-\frac{y+s^2}{2\sigma^2}}\, I_{\frac{n}{2}-1}\left(\frac{\sqrt{y}\,s}{\sigma^2}\right)$$

$F_Y(y)=1-Q_2\left(\frac{5}{1},\frac{\sqrt{y}}{1}\right)$

non-central $\chi$-square RV w/ 4 DoF

$s=5$

$\sigma=1$

where

$$I_\alpha(x)=\sum_{k=0}^{\infty}\frac{(\frac{x}{2})^{\alpha+2k}}{k!\,\Gamma(\alpha+k+1)}, \quad x \geq 0$$

if $\alpha=0,1,2,\cdots$ $=\frac{1}{\pi}\int_0^\pi e^{x\cos\theta}\cos n\theta\, d\theta$

the **noncentral chi-square pdf w/ n degrees of freedom** DoF DOF

$s^2$ the **noncentrality parameter of the distribution**

$$E\{Y\} = n\sigma^2 + x^2$$
$$E\{Y^2\} = 2n\sigma^4 + 4\sigma^2 x^2 + (n\sigma^2 + x^2)^2$$
$$E\{Y^2\} - E\{Y\}^2 = 2n\sigma^4 + 4\sigma^2 s^2$$

- Note, for all forms of $Y = \sqrt{\sum X_i^2}$ or $Y = \sum X_i^2$

w/ independent $X_i$'s with the same variance,
$f_Y(y)$ is a function of $x^2 = \sum_{i=1}^{n} \mu_i^2$ !

# Proper-complex Gaussian Distributions: Part II-0
Complex case

## ○ Complex-valued Gaussian random vector

- Def

Suppose that $\underline{X} \in \mathbb{R}^N$ and $\underline{Y} \in \mathbb{R}^N$ are real-valued random vectors. Then,

$$\underline{Z} \triangleq \underline{X} + j\underline{Y} \quad \in \mathbb{C}^N$$

is called **a complex-valued random vector** of length N if $\underline{X}$ and $\underline{Y}$ are jointly distributed.

"if" in definition ≡ iff.

- Remark

(i) A complex-valued random vector $\underline{Z} \in \mathbb{C}^N$ is completely characterized by the joint probability distribution function of $\mathrm{Re}\{\underline{Z}\}$ and $\mathrm{Im}\{\underline{Z}\}$.

(ii) A complex- 〃 〃 〃 〃 〃 〃 〃 〃 the joint characteristic function of $\mathrm{Re}\{\underline{Z}\}$ and $\mathrm{Im}\{\underline{Z}\}$

(iii) A complex-valued random vector with N-dimension is equivalent to a real-valued random vector of length $N$.

- Def

  A random vector $\underline{z} \triangleq \underline{x} + j\underline{y} \in \mathbb{C}^N$ is **a complex-valued Gaussian random vector** if the real-valued random vectors $\underbrace{\underline{x} \in \mathbb{R}^N \text{ and } \underline{y} \in \mathbb{R}^N}_{\text{abuse of notation.}}$ are jointly Gaussian

$\underline{z}$ length N complex Gaussian

$\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \sim N\left( \begin{bmatrix} \mu_{\underline{x}} \\ \mu_{\underline{y}} \end{bmatrix}, \begin{bmatrix} C_{\underline{x}\underline{x}} & C_{\underline{x}\underline{y}} \\ C_{\underline{y}\underline{x}} & C_{\underline{y}\underline{y}} \end{bmatrix} \right)$ real 2N Gaussian RV, N×N

$C_{\underline{x}\underline{y}} = E\{(\underline{x}-\mu_{\underline{x}})(\underline{y}-\mu_{\underline{y}})^T\}$

$C_{\underline{x}\underline{y}} = C_{\underline{y}\underline{x}}^T$

$\left( \begin{array}{l} [C_{\underline{x}\underline{y}}]_{ij} = E\{(x_i-\mu_{x_i})(y_j-\mu_{y_j})\} \\ [C_{\underline{y}\underline{x}}]_{ji} = E\{(y_j-\mu_{y_j})(x_i-\mu_{x_i})\} \end{array} \right.$

- Theorem

  A complex-valued Gaussian random vector $\underline{z} \in \mathbb{C}^N$ is completely characterized by the **mean** vector $\mu_{\underline{z}} \triangleq E[\underline{z}]$, the **covariance** matrix $\mathrm{Cov}\{\underline{z}\} \triangleq E\{(\underline{z}-\mu_{\underline{z}})(\underline{z}-\mu_{\underline{z}})^H\}$, and the **pseudo-covariance** matrix $\widetilde{\mathrm{Cov}}\{\underline{z}\} \triangleq E\{(\underline{z}-\mu_{\underline{z}})(\underline{z}-\mu_{\underline{z}})^T\}$.

/proof/

We know that $\underline{z}$ is completely characterized by the characteristic function $\varphi_{\underline{w}}(\underline{u})$ of $\overbrace{\phantom{xx}}^{\text{abuse of notation!}}$

$$\underline{w} \triangleq \begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{Re}\{\underline{z}\} \\ \mathrm{Im}\{\underline{z}\} \end{bmatrix} \in \mathbb{R}^{2N}$$

and that $\varphi_{\underline{w}}(\underline{u})$ is determined by $E\{\underline{w}\}$ & $\mathrm{Cov}\{\underline{w}\}$.

Note that

$$\begin{aligned} \mathrm{Cov}\{\underline{z}\} &= E\left[\{(\underline{x}-\mu_x)+j(\underline{y}-\mu_y)\}\{(\underline{x}-\mu_x)^T-j(\underline{y}-\mu_y)^T\}\right] \\ &= (\mathrm{Cov}\{\underline{x}\}+\mathrm{Cov}\{\underline{y}\}) \\ &\quad + j(-\mathrm{Cov}\{\underline{x},\underline{y}\}+\mathrm{Cov}\{\underline{y},\underline{x}\}) \end{aligned}$$

and that

(i) real composite vector

$$\underline{w} = \begin{bmatrix} \mathrm{Re}\{\underline{z}\} \\ \mathrm{Im}\{\underline{z}\} \end{bmatrix}$$

vs.

(ii) complex augmented vector

$$\begin{bmatrix} \underline{z} \\ \underline{z}^* \end{bmatrix}$$

$$\widetilde{Cov}\{\underline{Z}\} = E[\{(\underline{X}-\mu_X)+j(\underline{Y}-\mu_Y)\}\{(\underline{X}-\mu_X)+j(\underline{Y}-\mu_Y)\}^T]$$
$$= (Cov\{\underline{X}\} - Cov\{\underline{Y}\}) + j(Cov\{\underline{X},\underline{Y}\} + Cov\{\underline{Y},\underline{X}\})$$

Since $C_{XX} = Cov\{\underline{X}\} = \frac{1}{2}\{Re(Cov\{\underline{Z}\}) + Re(\widetilde{Cov}\{\underline{Z}\})\}$

$C_{YY} = Cov\{\underline{Y}\} = \frac{1}{2}\{Re(Cov\{\underline{Z}\}) - Re(\widetilde{Cov}\{\underline{Z}\})\}$

$C_{XY} = Cov\{\underline{X},\underline{Y}\} = \frac{1}{2}\{-Im(Cov\{\underline{Z}\}) + Im\ \widetilde{Cov}\{\underline{Z}\}\}$

and

$C_{XY}^T = C_{YX} = Cov\{\underline{Y},\underline{X}\} = \frac{1}{2}\{Im(Cov\{\underline{Z}\}) + Im(\widetilde{Cov}\{\underline{Z}\})\}$

$$\therefore Cov\{\underline{W}\} = \begin{bmatrix} Cov\{\underline{X}\} & Cov\{\underline{X},\underline{Y}\} \\ Cov\{\underline{Y},\underline{X}\} & Cov\{\underline{Y}\} \end{bmatrix}$$

is determined by $Cov\{\underline{Z}\}$ and $\widetilde{Cov}\{\underline{Z}\}$

Also,

$E\{\underline{W}\} = \begin{bmatrix} E\{\underline{X}\} \\ E\{\underline{Y}\} \end{bmatrix} = \begin{bmatrix} Re\{\mu_Z\} \\ Im\{\mu_Z\} \end{bmatrix}$ is determined by $E\{\underline{Z}\} = \mu_Z$

QED.

Lemma.
proper iff
(i) $C_{XX} = C_{YY}$
&
(ii) $C_{XY} = -C_{YX}$

$E[(X_i-\mu_{X_i})(Y_j-\mu_{Y_j})^*]$
$= -E[(Y_i-\mu_{Y_i})(X_j-\mu_{X_j})^*]$

• Notation

$\underline{Z} \sim N^c(\underline{\mu}; C, \tilde{C})$ or $\underline{Z} \sim N_c(\underline{\mu}; C, \tilde{C})$

$\underline{Z} \sim \tilde{N}(\underline{\mu}; C, \tilde{C})$

mean, cov, pseudo-cov.

$E\{\underline{Z}\} \triangleq E\{\underline{X}\} + jE\{\underline{Y}\} = \mu_X + j\mu_Y \triangleq \mu_Z$

$Cov\{\underline{Z}\} = E\{(\underline{Z}-\mu_Z)(\underline{Z}-\mu_Z)^H\}$

$\widetilde{Cov}\{\underline{Z}\} = E\{(\quad)(\quad)^T\}$

$\begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} \sim N([\ ], [\ ])$

3

$\tilde{N}(\mu_Z, C_{ZZ}, \tilde{C}_{ZZ})$

complex Gaussian

$C_{XY} = C_{YX}^H$ always

$= -C_{YX}$ proper

$(\Leftrightarrow C_{XY} = -C_{XY}^H)$

conjugate anti-symmetric

$\Rightarrow [C_{XY}]_{ii} = 0$

$[C_{XY}]_{ij} = -[C_{XY}]_{j,i}^*$

(ex) $\begin{bmatrix} 0 & -1+j \\ 1+j & 0 \end{bmatrix}$

• Def

$\underline{Z} \sim \tilde{N}(\underline{\mu}; C, \tilde{C})$ is called **proper** Gaussian if the pseudo-covariance vanishes, i.e., $\tilde{C} = 0$

• $\underline{Z} \sim CN(\underline{\mu}, C)$

$N(\mu, C)$ $\quad CN(\mu; C, \tilde{C})$

$CN(\mu, C)$ ← proper-complex & normal

• Theorem

If $\underline{Z} \sim \tilde{N}(\underline{\mu}; C, 0)$ has $C > 0$, then

$f_{\underline{X},\underline{Y}}(\underline{x},\underline{y})$ where $\underline{Z} \triangleq \underline{X} + j\underline{Y}$

$$= \frac{1}{\pi^N \det(C)} \exp(-(\underline{z}-\underline{\mu})^H C^{-1}(\underline{z}-\underline{\mu}))$$

$\triangleq f_{\underline{Z}}(\underline{z})$

1-D $\frac{1}{\pi\sigma^2}\exp(-\frac{|z-\mu|^2}{\sigma^2})$ $\quad E\{Z\} = \mu$, $Var\{Z\} = \sigma^2 \to \frac{\sigma^2}{2}, \frac{\sigma^2}{2}$

• Example

$Z \sim \tilde{N}(\mu; C, 0) \in \mathbb{C}^1$ and $C > 0$. (real positive)

Then

$Cov\{X\} = \frac{C}{2}$, $\quad Cov\{XY\} = 0$

$Cov\{Y\} = \frac{C}{2}$,

$$\Rightarrow f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{C}{2}}}\exp\left(-\frac{(x-Re(\mu))^2}{2\cdot\frac{C}{2}}\right)$$

$\mathrm{Cov}\{Y\} = \frac{C}{2}$,

$$\Rightarrow f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}\sqrt{\frac{C}{2}}} \exp\left(-\frac{(x-\mathrm{Re}\{\mu\})^2}{2\cdot\frac{C}{2}}\right)$$

$$\times \frac{1}{\sqrt{2\pi}\sqrt{\frac{C}{2}}} \exp\left(-\frac{(y-\mathrm{Im}\{\mu\})^2}{2\cdot\frac{C}{2}}\right)$$

$$= \frac{1}{\pi\cdot C} \exp\left(-(z-\mu)^* C^{-1}(z-\mu)\right)$$

$$\triangleq f_Z(z) \qquad \underset{2\pi\cdot\frac{C}{2}}{\parallel} \qquad \underset{-\frac{1}{2\frac{C}{2}}}{\parallel}$$

3′

• /proof/

Let $Z \triangleq X + jY$, $\mu \triangleq \mu_X + j\mu_Y$

$\underline{W} \triangleq \begin{bmatrix} X \\ Y \end{bmatrix}$

Then, $E\{\underline{W}\} = \begin{bmatrix} \mu_X \\ \mu_Y \end{bmatrix}$ and

$$\mathrm{Cov}\{\underline{W}\} = \begin{bmatrix} \mathrm{Cov}\{X\} & \mathrm{Cov}\{X,Y\} \\ \mathrm{Cov}\{Y,X\} & \mathrm{Cov}\{Y\} \end{bmatrix} \triangleq \begin{bmatrix} A & B \\ B^T & D \end{bmatrix}$$

From
$$\tilde{C} := E\{(Z-\mu)(Z-\mu)^T\}$$
$$= E\{[(X-\mu_X) + j(Y-\mu_Y)][(X-\mu_X) + j(Y-\mu_Y)]^T\}$$
$$= (\mathrm{Cov}\{X\} - \mathrm{Cov}\{Y\}) + j\{\mathrm{Cov}(X,Y) + \mathrm{Cov}(Y,X)\}$$
$$= 0$$

we have $\mathrm{Cov}\{X\} = \mathrm{Cov}\{Y\}$ and $\mathrm{Cov}(X,Y) = -\mathrm{Cov}(Y,X)$.

i.e., $A = D$ and $B = -B^T$

$C = 2(A - jB)$.

So, the LHS is

$$f_{XY}(x,y) = \frac{1}{\sqrt{2\pi}^{2N}\sqrt{\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix}}} \times \exp\left(-\frac{1}{2}\begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}^T \begin{pmatrix} A & B \\ -B & A \end{pmatrix}^{-1} \begin{bmatrix} x-\mu_x \\ y-\mu_y \end{bmatrix}\right),$$

and the RHS is

$$f_{\underline{z}}(\underline{z}) = \frac{1}{\pi^N \det\{2(A-jB)\}} \exp\left(-(\underline{z}-\underline{\mu})^H \{2(A-jB)\}^{-1} (\underline{z}-\underline{\mu})\right)$$

We need to show

$$\begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix}^T \begin{pmatrix}A & B\\ -B & A\end{pmatrix}^{-1} \begin{pmatrix}\underline{x}\\ \underline{y}\end{pmatrix} = \underline{z}^H (A-jB)^{-1} \underline{z} \quad \cdots (*)$$

and

$$\det(A-jB) = \sqrt{\det\begin{pmatrix}A & B\\ -B & A\end{pmatrix}} \quad \text{where } B = -B^T \quad \cdots (**)$$

(i) From the matrix inversion lemma:

$$\begin{pmatrix}A & B\\ C & D\end{pmatrix}^{-1} = \begin{bmatrix}(A-BD^{-1}C)^{-1} & -A^{-1}B(D-CA^{-1}B)^{-1}\\ -D^{-1}C(A-BD^{-1}C)^{-1} & (D-CA^{-1}B)^{-1}\end{bmatrix}$$

we have

$$\begin{pmatrix}A & B\\ -B & A\end{pmatrix}^{-1} = \begin{bmatrix}(A+BA^{-1}B)^{-1} & -A^{-1}B(A+BA^{-1}B)^{-1}\\ A^{-1}B(A+BA^{-1}B)^{-1} & (A+B^{-1}AB)^{-1}\end{bmatrix}$$

Moreover

$$(A-jB)\left[(I+jA^{-1}B)(A+BA^{-1}B)^{-1}\right]$$

$$= (A+BA^{-1}B)(A+BA^{-1}B)^{-1} = I$$

So the LHS is $\quad \triangleq G = G^T$

$$\underline{x}^T G \underline{x} - \underline{x}^T A^{-1} B G \underline{y} + \underline{y}^T A^{-1} B G \underline{x} + \underline{y}^T G \underline{y}$$

and the RHS is

$$(\underline{x}^T - j\underline{y}^T)(I+jA^{-1}B) G (\underline{x}+j\underline{y}) = \left\{(\underline{x}^T + \underline{y}^T A^{-1}B) + j(\underline{x}^T A^{-1}B - \underline{y}^T)\right\}\left\{G\underline{x} + jG\underline{y}\right\}$$

$$= \left(\underline{x}^T G \underline{x} + \underline{y}^T A^{-1} B G \underline{x}\right) - \left(\underline{x}^T A^{-1} B G \underline{y} - \underline{y}^T G \underline{y}\right)$$

$$5'$$

$$+j\underbrace{\left(x^T G y + y^T A^{-1} B G y + x^T A^{-1} B G x - y^T G x\right)}$$

$= 0 \quad \because \quad z^H \underbrace{(A-jB)^{-1}}_{\text{Hermitian symmetrical}} z$ is real

(ii) To show

"If $A = A^T > 0$ and $B = -B^T$
then
$$\det(A - jB) = \sqrt{\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix}}\text{"}$$

Let

$$\Lambda \triangleq A - jB$$

$$\therefore \Lambda^T = A^T - jB^T = A + jB = A(I + jA^{-1}B)$$

$$\Lambda^T (A + BA^{-1}B)^{-1} = A \underbrace{(I + jA^{-1}B)(A + BA^{-1}B)^{-1}}_{(A-jB)^{-1}}$$

$$= A\Lambda^{-1}$$

$$\therefore \Lambda^{-1} = A^{-1}\Lambda^T (A + BA^{-1}B)^{-1}$$

Then,

$$\det(\Lambda^{-1}) = (\det(A))^{-1} \det(\Lambda) \left(\det(A + BA^{-1}B)\right)^{-1}$$

$$= (\det(\Lambda))^{-1}$$

$$\therefore \det(\Lambda) = \sqrt{\det(A)\det(A + BA^{-1}B)}$$

$$\stackrel{?}{=} \sqrt{\det\begin{pmatrix} A & B \\ -B & A \end{pmatrix}} \quad \text{w/ } A = A^T > 0 \ \& \ B = -B^T$$

See Lancaster! for "?" part

◦ $n=N$ $\quad t_1, t_2, \cdots, t_N$

- Let $\tilde{\underline{X}} = \begin{bmatrix} X_c(t_1) \\ X_c(t_2) \\ \vdots \\ X_c(t_N) \\ X_s(t_1) \\ \vdots \\ X_s(t_N) \end{bmatrix} = \begin{bmatrix} \underline{X_c} \\ \underline{X_s} \end{bmatrix}$, then the mean and covariance function are given by

$$E[\tilde{\underline{X}}] = 0$$

$$Cov[\tilde{\underline{X}}] = E\begin{bmatrix} \underline{X_c}\,\underline{X_c}^T & \underline{X_c}\,\underline{X_s}^T \\ \underline{X_s}\,\underline{X_c}^T & \underline{X_s}\,\underline{X_s}^T \end{bmatrix}$$ , respectively

where

$\{E[\underline{X_c}\,\underline{X_c}^T]\}_{ij} = R_{X_cX_c}(t_j - t_i) \triangleq A$ symmetric

$\{E[\underline{X_s}\,\underline{X_s}^T]\}_{ij} = R_{X_sX_s}(t_j - t_i) = R_{X_cX_c}(t_j - t_i) = A$

$\{E[\underline{X_c}\,\underline{X_s}^T]\}_{ij} = R_{X_cX_s}(t_j - t_i) \triangleq B$ skew symmetric

$\{E[\underline{X_s}\,\underline{X_c}^T]\}_{ij} = R_{X_sX_c}(t_j - t_i) = -R_{X_cX_s}(t_j - t_i)$
$= -B = B^T$

$$\therefore Cov[\tilde{\underline{X}}] = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

- ~~Let~~ $\underline{X} = \underline{X_c} + j\underline{X_s}$
~~If Gaussian~~, $\underline{X}$ and $A$ is invertible then the density function is given

- Def

$\underline{z} \sim \tilde{N}(\mu; C, \tilde{C})$ is called **a white Gaussian random vector w/ mean $\mu$** if $\tilde{C} = 0$ and there exists $\sigma^2$ such that

$$C = 2\sigma^2 I$$

Note that $Cov(Re\{z_i\}, Re\{z_j\}) = \sigma^2 \delta_{ij}$
$Cov(Re\{z_i\}, Im\{z_j\}) = 0$
$Cov(Im\{z_i\}, Im\{z_j\}) = \sigma^2 \delta_{ij}$

every real, imaginary parts are i.i.d.

- **Properties** of proper Gaussian random vectors

(i) $E[\underline{z}]$ and $Cov\{\underline{z}\}$ completely characterize the distribution. (Already proved.)

(ii) An affine transform of $\underline{z}$, e.g.,

$\underline{w} = A\underline{z} + \underline{b}$ $\quad A \in \mathbb{C}^{M \times N}$
$\quad\quad\quad\quad\quad\quad \underline{b} \in \mathbb{C}^{M \times 1}$

is **also proper** Gaussian with mean
$E[\underline{w}] = A\mu_z + \underline{b}$ and Covariance
$Cov[\underline{w}] = AC_{zz}A^H$

(iii) Any **linear combination** of $z_i$'s becomes a circularly symmetric complex Gaussian random variable

$w = \underline{a}^H \underline{z}$

w.r.t. $\underline{a}^H \mu_z$

i.e.,
$E\{Re[w]\} = Re[\underline{a}^H \mu_z]$
$E\{Im[w]\} = Im[\underline{a}^H \mu_z]$
$Cov[Re\{w\}, Im\{w\}] = 0$
$Cov[Re\{w\}] = Cov[Im\{w\}]$.