2023년 4월 4일 화요일 12:34

CM12-230404Tue

Announcements

1. Midterm exam will be taken from 19:00-22:00 on Thursday Apr. 13th at LG102.
2. HW#9 is assigned.
3. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1.

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process

10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

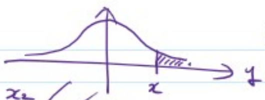
Review

1. 2-D
   a. Marcum's Q function
2. 2m-D
   a. generalized Marcum's Q function
3. n-D
   a. chi-square distribution for DOF n

Preview

1. proper-complex Gaussian random vector
   a. CDF of norm
   b. a complex random process
   c. why?
   d. a complex-valued vector random process
   e. widely-linear processing
2. Time average and ergodicity

(i) $Q(x) \triangleq \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$

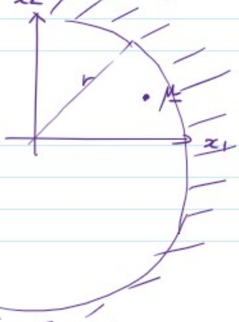


$X \sim N(\mu, \sigma^2)$ $\quad Pr(X \geq r) = Q\left(\frac{r-\mu}{\sigma}\right)$

(ii) $\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix}\right)$

$\alpha = \sqrt{\mu_1^2 + \mu_2^2}$

$Pr\left(\sqrt{X_1^2 + X_2^2} \geq r\right) = Q\left(\frac{\alpha}{\sigma}, \frac{r}{\sigma}\right)$



(iii) $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{2m} \end{bmatrix} \sim N(\underline{\mu}, \sigma^2 I)$ $\quad Pr(\|\underline{X}\| \geq r) = Q_m\left(\frac{\alpha}{\sigma}, \frac{r}{\sigma}\right)$

$\|\underline{x}\|_2 = \|\underline{x}\| = \sqrt{x_1^2 + \cdots + x_n^2}$

(iv) $\underline{X} = \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} \sim N(\underline{\mu}, \sigma^2 I)$ $\quad Y = \|\underline{X}\|^2$

Chi-square: central, non-central

5

$C = \sigma^2 I$

$C = 2\sigma^2 I$

white

Examples (proper Gaussian)

$Z \sim CN(0, 2\sigma^2)$

(i) When $Z \sim \tilde{N}(0, 2\sigma^2, 0)$ the joint density of X and Y ($Z = X + jY$) is given by

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) \equiv f_Z(z) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|z|^2}{2\sigma^2}\right)$$

(ii) In (i), Q1. $Pr(Re\{Z\} \geq a) = Q\left(\frac{a}{\sigma}\right)$ $\quad$ Q1.

(iii) In (i): Q2. $Pr(|Z| \geq b) = Q\left(0, \frac{b}{\sigma}\right)$

(iii) In (i), Q1. $\Pr(\mathrm{Re}\{z\} \ge a) = Q\left(\frac{a}{\sigma}\right)$

(iii) In (i), Q2 $\Pr(|z| \ge b) = Q\left(0, \frac{b}{\sigma}\right)$

(iv) When $\underline{z} \sim \tilde{N}(\underline{\mu}; 2\sigma^2 I, 0)$, $\underline{z} \sim CN(\underline{\mu}, 2\sigma^2 I)$

$\underline{z} \in \mathbb{C}^N$

5-2

Q3 $\Pr(\|\underline{z} - \underline{\mu}\| \ge \|\underline{z} - \underline{c}\|)$

$$= \Pr(\|\underline{z}-\underline{\mu}\|^2 \ge \|\underline{z}-\underline{c}\|^2)$$

$$= \Pr(\|\underline{z}\|^2 - 2\mathrm{Re}(\underline{\mu}^H \underline{z}) + \|\underline{\mu}\|^2 \ge \|\underline{z}\|^2 - 2\mathrm{Re}(\underline{c}^H \underline{z}) + \|\underline{c}\|^2)$$

$$= \Pr\left(\boxed{\mathrm{Re}\{(\underline{c}-\underline{\mu})^H \underline{z}\}} \ge \frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2}\right)$$

real

$$= \Pr\left(\mathrm{Re}\{w\} \ge \frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2}\right)$$

Note that the complex Gaussian random variable $w \triangleq (\underline{c}-\underline{\mu})^H \underline{z}$ has

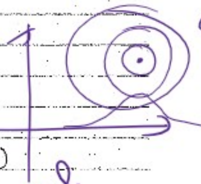
real Gaussian R.V.

$$E[w] = (\underline{c}-\underline{\mu})^H \underline{\mu}$$

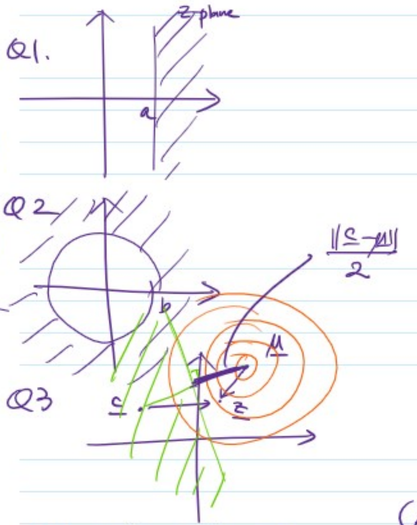
$$\mathrm{Cov}[w] = (\underline{c}-\underline{\mu})^H (2\sigma^2 I)(\underline{c}-\underline{\mu})$$

$$= Q\left(\frac{\frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2} - (\mathrm{Re}(\underline{c}^H \underline{\mu}) - \|\underline{\mu}\|^2)}{\sigma \|\underline{c}-\underline{\mu}\|}\right) \qquad \frac{\|\underline{c}\|^2 - 2\mathrm{Re}(\underline{c}^H\underline{\mu}) + \|\underline{\mu}\|^2}{2}$$

$$= Q\left(\frac{\frac{\|\underline{c}-\underline{\mu}\|^2}{2}}{\sigma\|\underline{c}-\underline{\mu}\|}\right) = Q\left(\frac{\|\underline{c}-\underline{\mu}\|}{2\sigma}\right)$$



Q1.



$(AB)^T = B^T A^T$, $(AB)^H = B^H$

ex/ $|z|^2 = zz^* = z^* z$

$\rightarrow \|\underline{z}\|^2 = \underline{z}^H \underline{z}$. $(\underline{a}^H \underline{b})^* = (\underline{a}^H \underline{b})^H = \underline{b}^H \underline{a}$

AB

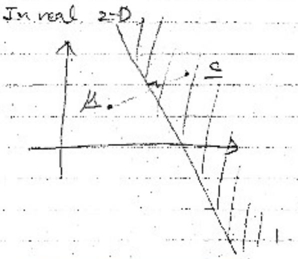
$z + z^* = 2\mathrm{Re}$

$$\|\underline{z}-\underline{\mu}\|^2 = (\underline{z}-\underline{\mu})^H(\underline{z}-\underline{\mu})$$
$$= (\underline{z}^H - \underline{\mu}^H)(\underline{z}-\underline{\mu})$$
$$= \underline{z}^H\underline{z} - \underline{z}^H\underline{\mu} - \underline{\mu}^H\underline{z} + \underline{\mu}^H\underline{\mu}$$
$$= \|\underline{z}\|^2 \underbrace{- 2\mathrm{Re}\{\underline{z}^H\underline{\mu}\}} + \|\underline{\mu}\|^2$$

$\mathrm{Re}\{a\} + \mathrm{Re}\{b\} = \mathrm{Re}\{a+b\}$

6

cf) In real 2-D



(v) In (iv), $\Pr(\|\underline{z}\| \ge c)$

$$= Q_N\left(\frac{\|\underline{\mu}\|}{\sigma}, \frac{c}{\sigma}\right)$$

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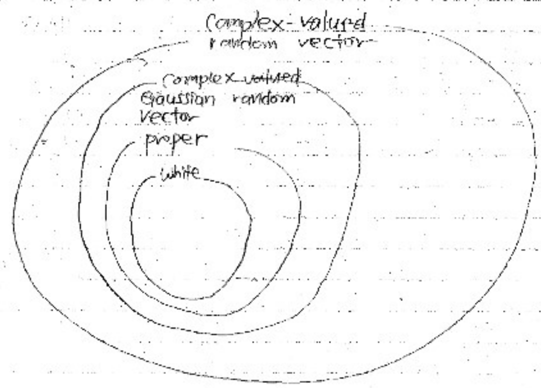


Baek Soo's problem 9/23/02

Q When $z \sim \tilde{N}(\mu, C, 0) \in \mathbb{C}^N$
find

$$Pr(\|z-\mu\| \ge \|z-p\|)$$

sol

$$Pr(\|z-\mu\|^2 \ge \|z-p\|^2)$$

$$= Pr\left(Re\{\underbrace{(p-\mu)^H z}_{\triangleq w}\} \ge \frac{\|p\|^2-\|\mu\|^2}{2}\right)$$

$$E\{w\} = (p-\mu)^H \mu$$

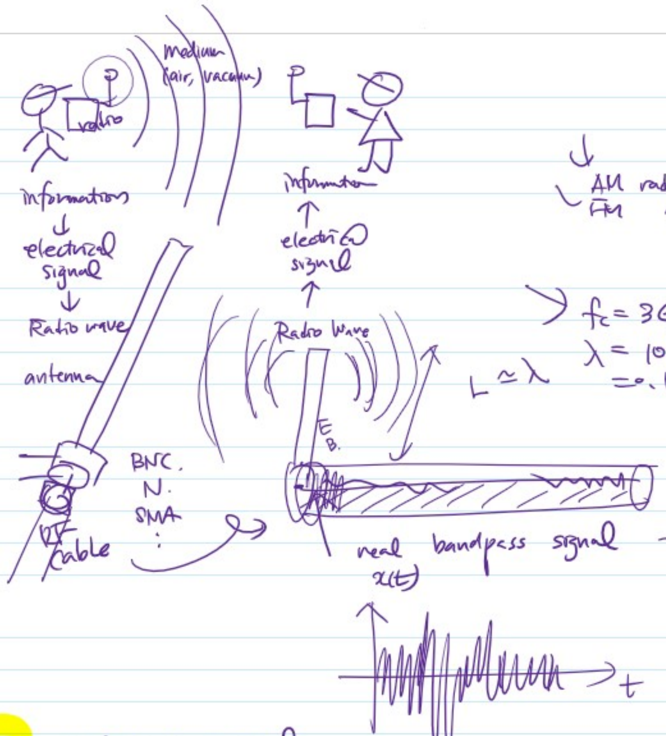
$$cov\{w\} = (p-\mu)^H C (p-\mu)$$

$$= Pr\left(Re\{w\} \ge \frac{\|p\|^2-\|\mu\|^2}{2}\right)$$

$$= Q\left(\frac{\frac{\|p\|^2-\|\mu\|^2}{2} - 2Re\{(p-\mu)^H \mu\}}{\sqrt{\frac{1}{2}(p-\mu)^H C (p-\mu)}}\right)$$

$\|p\|^2 - 2Re(p^H\mu) + \|\mu\|^2$

$$= Q\left(\frac{\|p-\mu\|^2}{\sqrt{2(p-\mu)^H C (p-\mu)}}\right)$$



AM radio 535 kHz − 1605 kHz
FM 〃 88 MHz − 108 MHz
$f_c = 30$ MHz
100x
$f_c = 3$ GHz
$\lambda = 10$ cm $= 0.1$ m
$\lambda = 10$ m
complex conjugate symmetry

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

real

$x(t)$ is real $\Leftrightarrow X(f) = X^*(-f)$

D $x(t)$ real bandpass signal

$X(f) = X^*(-f), \forall f$

complex baseband signal

$x(t) = \text{------} = Re\{x_c(t) e^{j2\pi f_c t}\}$

$x(t) = \cdots\cdots = \mathrm{Re}\{x_c(t)\, e^{j2\pi f_c t}\}$ ← signal

□

$Y(t) = x(t) + N(t)$ (real bp, real bp random process)

$N(t) = \cdots\cdots = \mathrm{Re}\{N_c(t)\, e^{j2\pi f_c t}\}$ — $N_c(t)$: complex baseband

Term.

- $x_c(t)$ is ~~the~~ a complex baseband equivalent to $x(t)$.
- $N_c(t)$ " " " " $N(t)$

□ Suppose that $N(t)$ (real bp) is a WSS Gaussian RP.

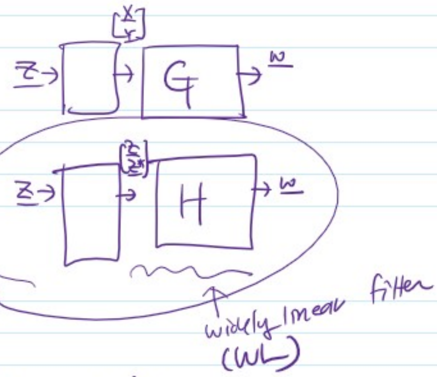
Q. $N_c(t)$?

A. (i) complex Gaussian
(ii) <u>proper</u> " "

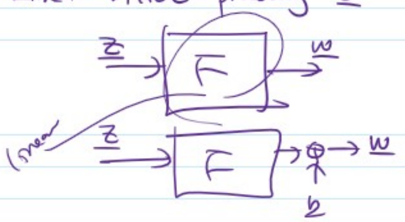
□ $\underline{z} \in \mathbb{C}^N$ $\quad \underline{z} = \underline{x} + j\underline{y} \quad \underline{x}, \underline{y} \in \mathbb{R}^N$

(i) $\begin{bmatrix} \underline{x} \\ \underline{y} \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\{\underline{z}\} \\ \mathrm{Im}\{\underline{z}\} \end{bmatrix} \in \mathbb{R}^{2N}$ — real composite vector

(ii) $\begin{bmatrix} \underline{z} \\ \underline{z}^* \end{bmatrix} = \begin{bmatrix} \underline{x} + j\underline{y} \\ \underline{x} - j\underline{y} \end{bmatrix}$ — complex augmented vector



□ Linear filter processing $\underline{z}$



$\underline{w} = F\underline{z}$ $\quad \min_F E\{\|\underline{z} - \underline{d}\|^2\}$

$\underline{w} = F\underline{z} + \underline{b}$ $\quad \min_{F, \underline{b}} \cdots$