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CM15-230418Tue

Announcements

1. Midterm exam was taken .
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
   Passwords are …
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process
10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

a time function

a scalar

$$X(t) \rightarrow \boxed{A[\ ]} \rightarrow A[X(t)]$$

a RV

$$A[x(t)] = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

Review

1. time average operator, time-autocorrelation function
   a. The time-average of a constant function is the constant.
2. … of a finite-energy signal/RP, a finite-power signal/RP
3. convergence of a function, convergence of a sequence
4. 4 modes on the convergence of a sequence of random variables
   a. mean-square convergence
5. Can we obtain a statistical average from a time average? Why do we need that?
6. Time average and stationarity?

(ex) $x(t)$

$$y(t) \triangleq X(t) X^*(t-\tau) \rightarrow \boxed{A[\ ]} \rightarrow A[X(t)X^*(t-\tau)] = \boxed{R}_{XX}(\tau)$$

cf) $R_{XX}(\tau) \triangleq E[X(t)X^*(t-\tau)]$

(ex) $|X(t)|^2 \rightarrow \boxed{A[\ ]} \rightarrow P_{XX} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |X(t)|^2 dt$

Preview

1. Def. of ergodicity and an ergodic random process
   a. Existence
2. Prove or assume?
   a. if not stationary not ergodic? Yes? No?
   b. "Theorem. If X(t) is stationary and ergodic in the XX sense, then….

- Def. The time average or the mean value of a random process $X(t)$ is defined as the limit of a stochastic integral:

$$\bar{X} \triangleq \mathbf{A}[X(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)dt.$$

– Q1. What is $\bar{X}$?

– A1. It is a random variable.

Consider the underlying probability space, a random process $X_t(s)$, and a time-average system.

Then, the output of the system is

$$\mathbf{A}[X_t(s)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X_t(s)dt = \bar{X}(s)$$

which is a function of s. Hence, it is a random variable.

– Note that we have uncountably infinite number of random variables

$$\frac{1}{2T}\int_{-T}^{T} X_t(s)dt$$

parameterized by T.

Note also that we claim the convergence of the sequence of random variables to a random variable $\bar{X}$ in some sense as T tends to infinity.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{2x^2-1}{x^2+x}$$

if → this limit exists then

for any $x_n \to \infty$

$$\lim \frac{2x_n^2-1}{\ldots}$$

$$\frac{1}{2T}\int_{-T}^{T} X(t)dt$$

parameterized by T.

Note also that we claim the convergence of the sequence of random variables to a random variable $\bar{X}$ in some sense as T tends to infinity.

for any $x_n \to \infty$

This limit exists then

$$\lim_{n\to\infty} \frac{2x_n^2 - 1}{x_n^2 + x_n} \to \text{some limit}$$

– For the convergence of a sequence of random variables, among 4 types/modes of convergence, we here only consider the convergence in the mean-square (MS) sense or the mean-square (MS) convergence defined as

$$X_n \to X \text{ in the MS sense } \Leftrightarrow \mathbf{E}[|X_n - X|^2] \to 0 \text{ as } n \to \infty.$$

$*$ The MS convergence is often denoted by

$$\underset{n\to\infty}{\text{l.i.m.}}\ X_n = X,$$

which is read the limit in the mean of X_n is X.

– Q2. What is $\mathbf{E}[X]$?

– A2. It is a number.

$A[X(t)]$

$5 = 5$

$\Leftrightarrow 5 - 5 = 0$

$\Leftrightarrow |5-5|^2 = 0$

$\Leftrightarrow E\{|5-5|^2\} = 0$

$P(X = Y) = 1$

$P(X - Y = 0) = 1$

$P(|X-Y|^2 = 0) = 1$

$\overset{?}{\Leftrightarrow} E\{|X-Y|^2\} = 0.$

$X_n \simeq X$

f.s.l. n

$E\{|X_n - X|^2\} \to 0$

• Def. The time autocorrelation function of a random process $X(t)$ is defined as

$$\begin{aligned}\mathcal{R}_{XX}(\tau) &\triangleq \mathbf{A}[X(t+\tau)X^*(t)] \\ &= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t+\tau)X^*(t)dt.\end{aligned}$$

– Q1. What is $\mathcal{R}_{XX}(\tau)$?

– A1. It is a random variable parameterized by τ.

Consider the underlying probability space, a random process $X_t(s)$, and a time-average system.

Then, the output of the system is

$$\mathbf{A}[X_{t+\tau}(s)X_t^*(s)] = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X_{t+\tau}(s)X_t^*(s)dt = \mathcal{R}_{XX,\tau}(s)$$

which is a function of s parameterized by τ.

Hence, it is a random variable parameterized by τ.

– Again note that we have uncountably infinite number of random variables

$$\frac{1}{2T}\int_{-T}^{T} X(t+\tau)X^*(t)dt$$

parameterized by T.

Again note also that we claim the convergence of the sequence of random variables to a random variable $\mathcal{R}_{XX}(\tau)$ in some sense as T tends to infinity.

$f(x) = \sin \pi x$

$\lim_{x\to\infty} f(x) =$ Not exist

$\lim_{n\to\infty} f(n) = 0$

– Q2. What is $\mathbf{E}[\mathcal{R}_{XX}(\tau)]$?

– A2. It is a function of τ.

- Q. If $X(t)$ is stationary (and the expectation can be brought inside the integrals), then what can we say about

1. $\mathbf{E}[\bar{X}]$ and
2. $\mathbf{E}[\mathcal{R}_{XX}(\tau)]$?

– A1. The statistical/ensemble average of the time average of a random process equals the statistical mean path that is a constant function, i.e.,

$$\mathbf{E}[\bar{X}] \triangleq \mathbf{E}\{\mathbf{A}[X(t)]\} = \mathbf{A}\{\mathbf{E}[X(t)]\} = \mu_X$$

$\because$ stationary

– A2. The statistical/ensemble average of the time autocorrelation function equals the autocorrelation function, i.e.,

$\because$ stationary

$$\mathbf{E}[\mathcal{R}_{XX}(\tau)] \triangleq \mathbf{E}\{\mathbf{A}[X(t+\tau)X^*(t)]\} = \mathbf{A}\{\mathbf{E}[X(t+\tau)X^*(t)]\} = R_{XX}(\tau).$$

– The stationarity itself does not directly link the time average and statistical average of a RP.

– The equalities above still require the ensemble averages of the time averages to link the time average and statistical average.

– Example. $P(\{H\}) = 1/2, P(\{T\}) = 1/2$. $X_t(H) = 1, \forall t$, $X_t(T) = -1, \forall t$

$$\frac{m}{m+n}$$

* Ensemble average: $X(t)$ is stationary and has $\mathbf{E}[X(t)] = 0, \forall t$ and $R_{XX}(t+\tau, t) = R_{XX}(\tau) = 1, \forall t, \tau$.
* Time average: $\mathbf{A}[X_t(H)] = 1, \forall t$ and $\mathbf{A}[X_t(T)] = -1, \forall t$, and $\mathcal{R}_{XX}(t+\tau, t) = \mathcal{R}_{XX}(\tau) = 1, \forall \tau$.
* Means: $\mathbf{E}[\bar{X}] = \mathbf{E}[X(t)] = 0$. $\leftarrow$ ensemble average of time average
* Autocorrelations: $\mathcal{R}_{XX}(\tau) = R_{XX}(\tau), \forall \tau$. $\leftarrow$ no ensemble average of time average

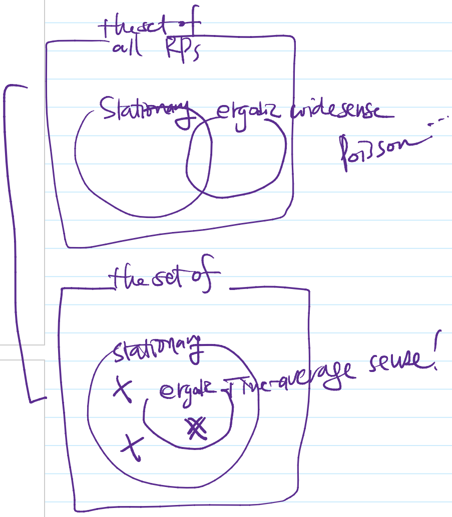
- Def. A random process $X_t(s)$ is ergodic in some sense if a time average is equal to the corresponding statistical average.
  - We are usually forced to work with only one sample function in the real world and therefore must, like it or not, derive mean value, correlation functions, etc., from the time function.
    - Ergodicity is a very restrictive property.
    - It may be difficult to prove that it constitutes a reasonable assumption in any physical situation.
    - It is often assumed to simplify problems.
  - There are many levels of ergodicity.
  1. Which average? We may want

     $E[X(t+\tau)X^*(t)]$

     $$\mathbf{A}[X(t)] = \mathbf{E}[X(t)]$$
     $$\mathcal{R}_{XX}(\tau) = \cancel{R_{XX}(\tau)}, \forall\tau,$$

     $R_{XX}(t+\tau, t)$

     which are called ergodic in the mean function and ergodic in the auto-correlation function, respectively.
  2. Which type/mode of convergence? In this course, we focus on the simplest, MS ergodicity only.
     - mean-square ergodic in the mean function = mean ergodic in the MS sense
     - mean-square ergodic in the auto-correlation function = auto-correlation ergodic in the MS sense



  3. Stark and Woods, p. 443: "The property of MS ergodicity occurs when the random process decorrelates sufficiently rapidly with time shift so that the time-average in question looks like the average of many almost uncorrelated random variables, which in turn–by appropriate forms of the weak law of large numbers–will converge to the appropriate ensemble average."
     - Divide the integration interval into many subintervals, and convert the integral into the arithmetic average of almost uncorrelated random variables.

     "If the random process stays correlated over infinitely long time intervals, then we would not expect such behavior, and indeed ergodicity may not hold."

- Def. Multiple random process are jointly ergodic in some sense if a time average is equal to the corresponding statistical average.