CM16-230418Thu

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Announcements

1. Midterm exam was taken .
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

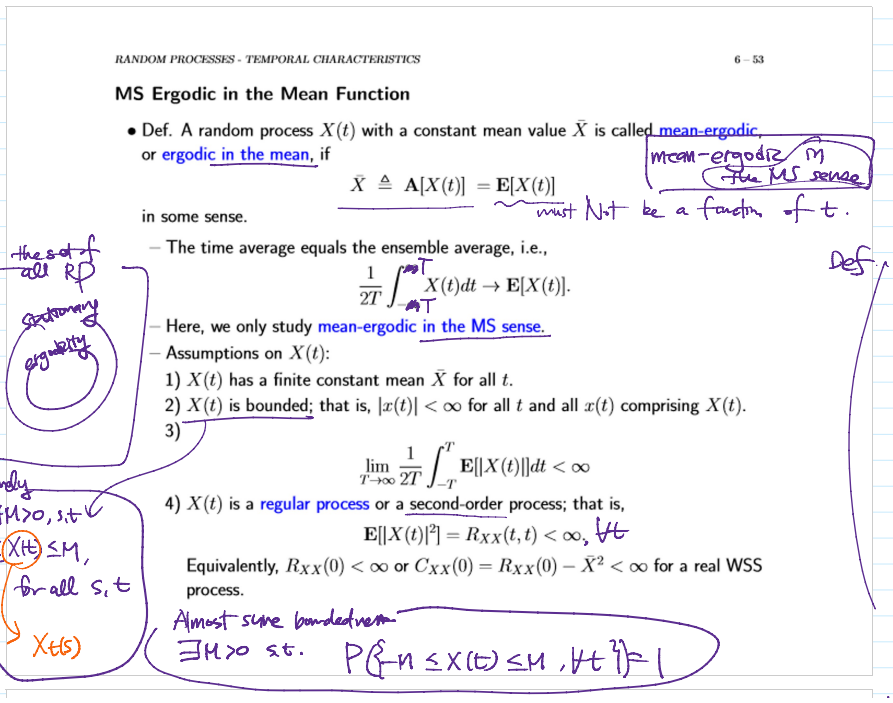
9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process

10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. Ergodicity
   a. wide sense vs. narrow sense
   b. Ergodicity is a very restricitve stationarity.
   c. Assumed

Preview



RANDOM PROCESSES - TEMPORAL CHARACTERISTICS 6 – 53

MS Ergodic in the Mean Function

- Def. A random process $X(t)$ with a constant mean value $\bar{X}$ is called mean-ergodic, or ergodic in the mean, if

$$\bar{X} \triangleq \mathbf{A}[X(t)] = \mathbf{E}[X(t)]$$

in some sense.

mean-ergodic in the MS sense

must Not be a function of t.

– The time average equals the ensemble average, i.e.,

$$\frac{1}{2T}\int_{-T}^{T} X(t)dt \to \mathbf{E}[X(t)].$$

– Here, we only study mean-ergodic in the MS sense.

– Assumptions on $X(t)$:

1) $X(t)$ has a finite constant mean $\bar{X}$ for all t.

2) $X(t)$ is bounded; that is, $|x(t)| < \infty$ for all t and all $x(t)$ comprising $X(t)$.

3)

$$\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\mathbf{E}[|X(t)|]dt < \infty$$

4) $X(t)$ is a regular process or a second-order process; that is,

$$\mathbf{E}[|X(t)|^2] = R_{XX}(t,t) < \infty, \forall t$$

Equivalently, $R_{XX}(0) < \infty$ or $C_{XX}(0) = R_{XX}(0) - \bar{X}^2 < \infty$ for a real WSS process.

the set of all RP

stationary & ergodicity

Surely $\exists M>0$, s.t. $-M \le X(t) \le M$, for all s, t

$X(t,s)$

Almost sure bounded-ness

$\exists M>0$ s.t. $P\{-M \le X(t) \le M, \forall t\} = 1$

Def. Given a seq. of RVs $\{X_n\}_{n\in\mathbb{N}}$ & a limit RV X,

$X_n \to X$ in the MS sense

if

$E\{|X_n - X|^2\} \to 0$ as $n\to\infty$

Denoted by

$\lim_{n\to\infty} X_n = X$

$X(t)$

X(t,s)

Almost sure boundedness
$\exists M>0$ s.t. $P(\{-M \le X(t) \le M, \forall t\}) = 1$

Remarks: The second and third assumptions are necessary to allow the interchange of operations of expectation and integration subsequently needed.

- Q. What is a condition for a real random process to be mean-ergodic in the MS sense?
- A.
  - Given a random process $X(t)$, define a random variable

$$A_X \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} X(t)\,dt.$$

  * The mean of the random variable A_X is

$$\bar{A}_X = \mathrm{E}[A_X] = E\left\{\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} X(t)dt\right\} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\bar{X}\,dt = \bar{X} \quad (6.2\text{-}1)$$

  * Chebychev's inequality:

$$P(|A_X - \bar{A}_X| \le \epsilon) \ge 1 - \sigma^2_{A_X}/\epsilon^2 \quad \text{any } \epsilon > 0$$

tells us that $A_X = \bar{A}_X$ with probability 1 if the variance of A_X is zero. Hence, we want to find (sufficient) conditions under which the variance is zero.

Def. A CT RP X(t) is regular if $E\{|X(t)|^2\} < \infty, \forall t$
a second-order process

$P(|A_{X_T} - \bar{X}| < \varepsilon) \ge 1 - \frac{\sigma^2_{A_{X_T}}}{\varepsilon^2}$, $\forall t$

Show $\sigma^2_{A_{X_T}} \to 0$ as $T\to\infty$

$\lim_{T\to\infty} P(|A_{X_T} - \bar{X}| < \varepsilon) = 1$, $\forall \varepsilon$

$\varepsilon_n \downarrow 0$

If A_n decreasing then $\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n)$

$\therefore \lim_{n\to\infty} P(|-| \le \varepsilon_n) = P(|-| \le 0) = 1$

$\Leftrightarrow P(-=0) = 1$

$\Pr(\lim_{T\to\infty} A_{X_T} = \bar{X}) = 1$

$f(x)$ is continuous @ x_0 $\Leftrightarrow$ $\lim_{x\to x_0} f(x) = f(\lim_{x\to x_0} x)$

$\mathrm{Var}(A_X)$

□ the Markov inequality

Def. A RV is nonnegative if $P(X \ge 0) = 1$

1-D scatter plot

Thm. Given a non-negative RV X w/ $\mu_X \triangleq E[X] < \infty$,

$P(X \ge a) \le \frac{\mu_X}{a}$

$F_X(x) \triangleq P(X \le x)$

$1 - F_X(x)$

$\frac{x}{a} \ge u(x-a), \forall x \ge 0$

$P(A) = E\{1_A\} = 1 \times P(A) + 0 \times P(A^c)$

$1 \times P(X \ge a) + 0 \times P(X < a)$

(i) $\Rightarrow P\{\frac{X}{a} \ge u(X-a)\} = 1$ if X is a non-negative RV.

(ii) $E\{\frac{X}{a}\} \ge E\{u(X-a)\}$

$P(X \ge a) \le \frac{\mu_X}{a}$

□ Chebychev Inequality

$P(X \ge a) \le \frac{\mu_X}{a}$ $\quad E[X]$

$X = Y^2 \ge 0$ w.p. 1 $\quad P(X \ge 0) = 1$

$X \ge 0$ w.p. 1

~~$Y \ge 0$ w.p. 1~~

$P(Y^2 \ge a) \le \frac{E\{Y^2\}}{a}$ $\quad E\{Y\} < \infty$, $E\{Y^2\} < \infty$

$P((Y-\mu_Y)^2 \ge \varepsilon^2) \le \frac{\mathrm{Var}\{Y\}}{\varepsilon^2}$

$\Leftrightarrow P(|Y-\mu_Y| \ge \varepsilon) \le \frac{\mathrm{Var}\{Y\}}{\varepsilon^2}$

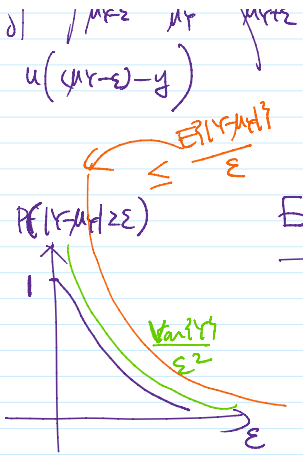
$\|x - a\|$ from a

$u(y-(\mu_Y+\varepsilon))$ to x

distance

$u((\mu_Y-\varepsilon)-y)$

$(y-\mu_Y)^2$



$$\frac{(y-\mu_Y)^2}{\varepsilon^2} \geq u(|y-\mu_Y|-\varepsilon), \forall y$$

$$E\left\{\left(\frac{Y-\mu_Y}{\varepsilon}\right)^2\right\} \geq E\{u(|Y-\mu_Y|-\varepsilon)\}, \text{RHS}$$

$$\frac{Var[Y]}{\varepsilon^2} \geq P(|Y-\mu_Y|-\varepsilon \geq 0)$$

$$P(|Y-\mu_Y| \geq \varepsilon)$$

* The variance of the random variable A_X is

$$\sigma_{A_X}^2 = \mathbf{E}[(A_X - \bar{A}_X)^2] = E\left\{\left[\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}[X(t)-\bar{X}]dt\right]^2\right\}$$
$$= E\left\{\lim_{T\to\infty}\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\int_{-T}^{T}[X(t)-\bar{X}][X(t_1)-\bar{X}]dtdt_1\right\}$$
$$= \lim_{T\to\infty}\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\int_{-T}^{T}C_{XX}(t,t_1)dtdt_1 \qquad (6.2\text{-}2)$$

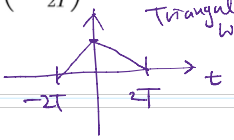
* For real regular WSS processes where
$C_{XX}(t,t_1) = C_{XX}(\tau), \tau = t_1 - t, dt_1 = d\tau,$

$$\sigma_{A_X}^2 = \lim_{T\to\infty}\left(\frac{1}{2T}\right)^2\int_{t=-T}^{T}\int_{\tau=-T-t}^{T-t}C_{XX}(\tau)d\tau dt$$
$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau \quad (i) \quad \text{iff}$$
$$\leq \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}|C_{XX}(\tau)|d\tau \quad (ii) \quad \text{if}$$

– A necessary and sufficient condition: MS ergodic in the mean iff

$$\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau \to 0 \text{ as } T\to\infty.$$

Triangular window !!



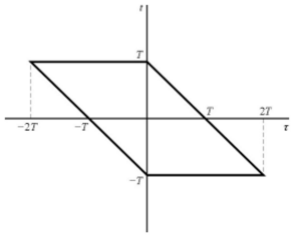


Figure 6.2-1 Closed region of integration

– A sufficient condition: MS ergodic in the mean if

$$\int_{-\infty}^{\infty}|C_{XX}(\tau)|d\tau < \infty.$$

As if "a finite-energy signal has zero power"

• Examples

1. $X(t) = A, \forall t$, A a random variable is SSS but not ergodic, not MS ergodic in the mean function.
2. $X(t) = R\cos(2\pi f_0 t + \Theta)$ with $f_0 \neq 0$, independent Rayleigh R and uniform Θ on $(0, 2\pi]$
   – It is a zero-mean SSS Gaussian random process.