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Announcements

1. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
2. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

1. Sorry, there are many corrections.

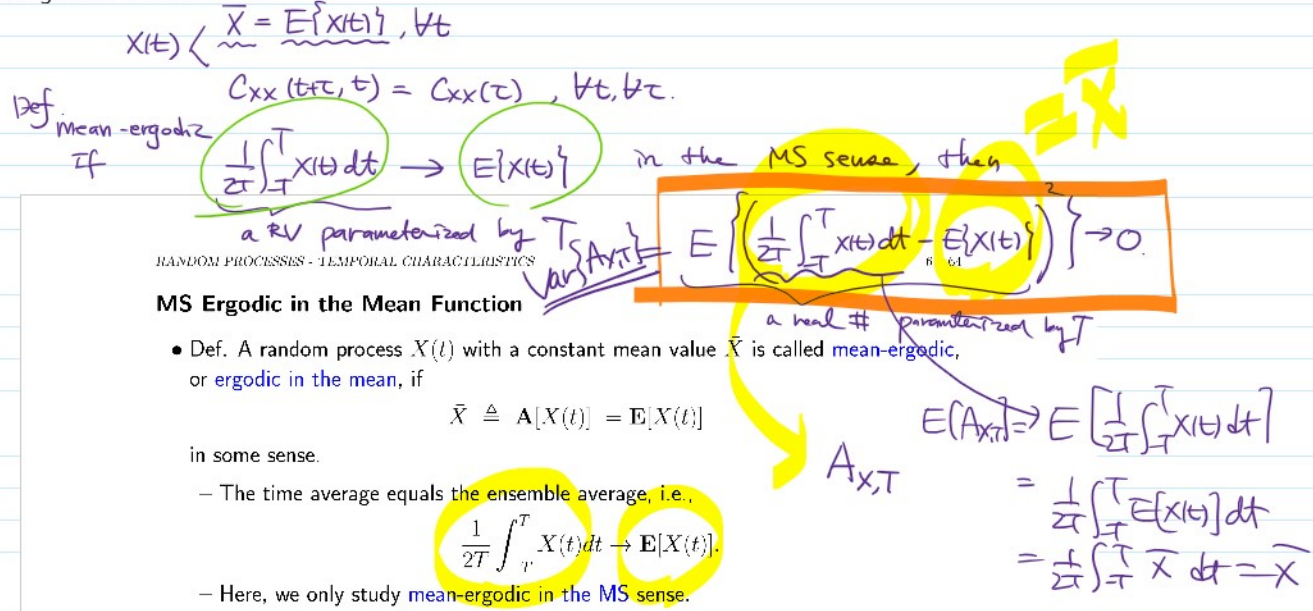
Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process

10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. ergodic in the MS sense



RANDOM PROCESSES - TEMPORAL CHARACTERISTICS 6 64

MS Ergodic in the Mean Function

- Def. A random process $X(t)$ with a constant mean value $\bar{X}$ is called mean-ergodic, or ergodic in the mean, if

$$\bar{X} \triangleq \mathbf{A}[X(t)] = \mathbf{E}[X(t)]$$

in some sense.

  - The time average equals the ensemble average, i.e.,

$$\frac{1}{2T}\int_{T} X(t)dt \to \mathbf{E}[X(t)].$$

  - Here, we only study mean-ergodic in the MS sense.
  - Assumptions on $X(t)$:
    1) $X(t)$ has a finite constant mean $\bar{X}$ for all t.
    2) $X(t)$ is bounded; that is, $|x(t)| < \infty$ for all t and all $x(t)$ comprising $X(t)$.
    3)

$$\lim_{T\to\infty} \frac{1}{2T}\int_{T} \mathbf{E}[|X(t)|]dt < \infty$$

    4) $X(t)$ is a regular process or a second-order process; that is,

$$\lim_{T\to\infty}\frac{1}{2T}\int_{T}^{T}\mathbf{E}[|X(t)|]dt<\infty$$

4) $X(t)$ is a regular process or a second-order process; that is,

$$\mathbf{E}[|X(t)|^2]=R_{XX}(t,t)<\infty.$$

Equivalently, $R_{XX}(0)<\infty$ or $C_{XX}(0)=R_{XX}(0)-\bar{X}^2<\infty$ for a real WSS process.

Remarks: The second and third assumptions are necessary to allow the interchange of operations of expectation and integration subsequently needed.

- Q. What is a condition for a real random process to be mean-ergodic in the MS sense?
- A.
  - Given a random process $X(t)$, define a random variable

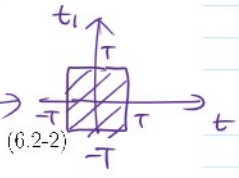
$$A_{X,T}\triangleq\frac{1}{2T}\int_{-T}^{T}X(t)dt.$$

  * The mean of the random variable $A_{X,T}$ is

$$\begin{aligned}\bar{A}_{X,T}&=\mathbf{E}[A_{X,T}]\\&=E\left\{\frac{1}{2T}\int_{-T}^{T}X(t)dt\right\}=\frac{1}{2T}\int_{-T}^{T}\bar{X}dt=\bar{X}\end{aligned}\qquad(6.2\text{-}1)$$

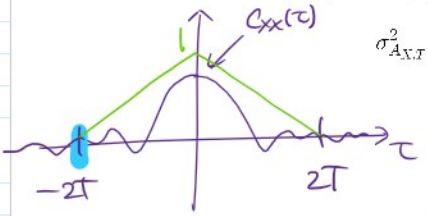
  * The variance of the random variable $A_{X,T}$ is

$$\begin{aligned}\sigma^2_{A_{X,T}}&=\mathbf{E}[(A_{X,T}-\bar{A}_{X,T})^2]=E\left\{\left[\frac{1}{2T}\int_{T}^{T}[X(t)-\bar{X}dt\right]^2\right\}\\&=E\left\{\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\int_{-T}^{T}[X(t)-\bar{X}][X(t_1)-\bar{X}]dtdt_1\right\}\\&=\left(\frac{1}{2T}\right)^2\int_{-T}^{T}\int_{-T}^{T}C_{XX}(t,t_1)dtdt_1\end{aligned}\qquad(6.2\text{-}2)$$



* For real regular WSS processes where $C_{XX}(t,t_1)-C_{XX}(\tau),\ \tau=t_1-t,\ dt_1=d\tau,$

$$\begin{aligned}\sigma^2_{A_{X,T}}&=\left(\frac{1}{2T}\right)^2\int_{t=-T}^{T}\int_{\tau=-T-t}^{T-t}C_{XX}(\tau)d\tau dt\\&=\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau\\&\le\frac{1}{2T}\int_{-2T}^{2T}|C_{XX}(\tau)\,d\tau\end{aligned}$$



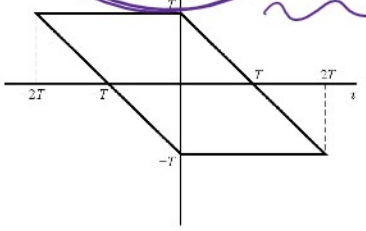
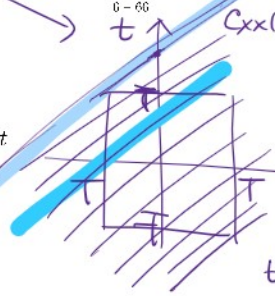


Figure 6.2-1 Closed region of integration



$t-t_1=2T$

$t=t_1+2T$

$t=t_1-2T$

$\int_{-\infty}^{\infty}|C_{XX}(\tau)|d\tau<\infty$

$\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)\,d\tau$

$\le\left|\ \ =\ \ \right|$

$\le\frac{1}{2T}\int_{-2T}^{2T}\left|\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)\right|d\tau$

$=\frac{1}{2T}\int_{-2T}^{2T}|\ C_{XX}(\tau)|\,d\tau$

recall $\int_{-\infty}^{\infty}|x(t)|^2dt<\infty$

$A\{|x(t)|^2\}\triangleq\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|x(t)|^2dt$

Summary. A WSS real RP $X(t)$ is mean ergodic in the MS sense (pdm...) iff

(i) $\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)\,d\tau\to 0$ as $T\to\infty$ ← triangular windowed covariance fct has 0 time average.

(ii) if $\int_{-\infty}^{\infty}|C_{XX}(\tau)|\,d\tau<\infty$ (← the covariance fct is absolutely integrable)

(i) iff $\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)\,d\tau \to 0$ as $T\to\infty$. time average.

(ii) if $\int_{-\infty}^{\infty}|C_{XX}(\tau)|\,d\tau < \infty$ (⇐ the covariance fn is absolutely integrable)

– A necessary and sufficient condition: MS ergodic in the mean iff

$$\frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|\tau|}{2T}\right)C_{XX}(\tau)d\tau \to 0 \text{ as } T\to\infty.$$

– A sufficient condition: MS ergodic in the mean if

$$\int_{-\infty}^{\infty}|C_{XX}(\tau)|d\tau < \infty.$$

- Examples

ex 1. $X(t)=A,\forall t$, A a random variable is SSS but not ergodic, not MS ergodic in the mean function.

ex 2. $X(t)=R\cos(2\pi f_0 t+\Theta)$ with $f_0\neq 0$, independent Rayleigh R and uniform Θ on $(0,2\pi]$

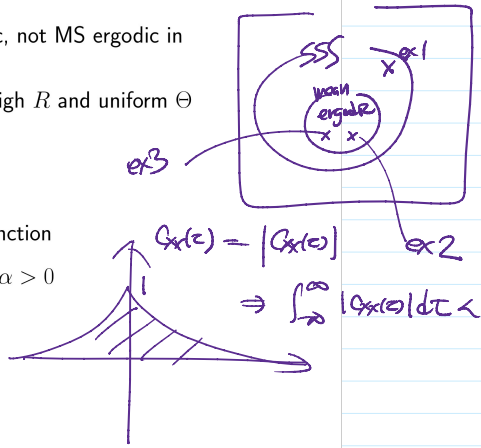
– It is a zero-mean SSS Gaussian random process.

– It is MS ergodic in the mean.

ex 3. A zero-mean WSS process $X(t)$ has an autocorrelation function

$$R_{XX}(\tau)=C_{XX}(\tau)=\exp(-2\alpha|\tau|) \quad \text{for } \alpha>0$$

Determine if $X(t)$ is mean-ergodic in the MS sense.
(solution)



Note that $C_{XX}(0)<\infty$ and $C_{XX}(\tau)\to 0$ as $|\tau|\to\infty$. Furthermore,

$$\begin{aligned} I &\triangleq \lim_{T\to\infty}\frac{1}{2T}\int_{-2T}^{2T}|C_{XX}(\tau)|\,d\tau \\ &= \lim_{T\to\infty}\frac{1}{2T}\,2\int_{0}^{2T}e^{-2\alpha\tau}d\tau \\ &= \lim_{T\to\infty}\frac{1}{2T}\,\frac{1}{\alpha}\,(1-e^{-4\alpha T}) \\ &= 0. \end{aligned}$$

Therefore, $\sigma_{A_X}^2=0$ and $X(t)$ is mean-ergodic in the MS sense.

Mean-Ergodic DT Random Process

- Random sequence $X[n]$
  - a sequence of values generated from a WSS random process $X(t)$ with constant mean $\bar{X}$ by sampling $2N+1$ times at periodic points in time from $-T$ to $+T$.
  - $X[n]$ is a discrete-time sequence of identically distributed RVs.
  - Samples are assumed to be statistically independent, which is approximately true in practice if samples are separated far enough in time.
- Def. A DT random process is called mean-ergodic if the time average of samples equals the statistical average in some sense, that is,

$$A_X \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} X[n] = \bar{X}, \quad \text{in some sense.}$$

  - Def. mean-ergodic in the MS sense
  - Lemma. mean-ergodid in the MS sense iff $X[n]$ is a regular WSS random process having

$$\sigma^2_{A_{X,N}} \triangleq \mathbf{E}[(A_{X,N} - \bar{A}_{X,N})^2] = \frac{1}{2N+1} \sum_{n=-2N}^{2N} \left(1 - \frac{|n|}{2N+1}\right) C_{XX}[n] \to 0 \quad \text{as } N\to\infty.$$

if $\sum_{n=-\infty}^{\infty} |C_{XX}[n]| < \infty$

mean ergodicity in MS sense < CT, DT RPs

$Y(t) = |X(t)|^2$ the instantaneous power of $X(t)$

MS Ergodic in the Power Function

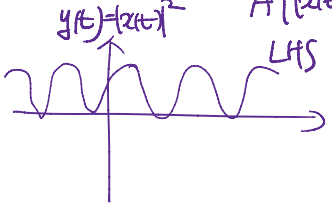
- Def. If

$$\mathbf{A}[|X(t)|^2] = \mathbf{E}[|X(t)|^2]$$

LHS

in the MS sense, then $X(t)$ is MS ergodic in power or ... in the power function.

- Example. $X(t) = R\cos(2\pi f_0 t + \Theta)$ with independent Rayleigh R and uniform Θ on $[0, 2\pi]$
  - It is a zero-mean SSS Gaussian random process.
  - It is MS ergodic in the mean.
  - It is NOT MS ergodic in power.



$A\{|x(t)|^2\} = \frac{r^2}{2}$

LHS $A\{|X(t)|^2\} = \frac{R^2}{2}$

MS Ergodic in Correlation Function

- Def. A stationary continuous process $X(t)$ with autocorrelation function $R_{XX}(\tau)$ is called autocorrelation-ergodic or ergodic in the autocorrelation in some sense if

$$\mathcal{R}_{XX}(\tau) = R_{XX}(\tau), \forall \tau.$$

  - Def. correlation-ergodic in the MS sense
  - Lemma. A real random process is correlation-ergodic in the MS sense if $W(t) = X(t+\lambda)X(t)$ is a real regular WSS random process satisfying $\int_{-\infty}^{\infty} |C_{WW}(\tau)| d\tau < \infty$
    - The condition for $X(t)$ to be autocorrelation-ergodic is the condition for $W(t)$ to be mean-ergodic.
    - Autocorrelation-ergodicity requires fourth-order moments of $X(t)$.
    - This requirement may be severe for general random processes, but fourth-order moments for Gaussian random processes are known in terms of only moments of order two or less...

- Def. Two processes $X(t)$ and $Y(t)$ may be called cross-correlation-ergodic or ergodic in the correlation function, if the time cross-correlation function is equal to the statistical cross-correlation function.

$\Leftarrow \int_{-\infty}^{\infty} |C_{WW}(\tau)| d\tau < \infty$

$\Leftrightarrow \frac{1}{2T}\int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{WW}(\tau) d\tau \to 0$

$W(t)$ is mean ergodic in the MS sense

$\Leftrightarrow X(t)^2$ " " " " " "

$W(t)$ is mean ergodic in the MS sense

$\Leftrightarrow X(t+\lambda)X(t)$ is " " " " " "

MS Ergodic in the Distribution Function

- This section is from Section 9.4 of H. Stark and J. W. Woods, *Probability, Random Processes, and Estimation Theory for Engineers*, 2nd ed.
- Def. Given a random process $X(t)$, the indicator processor with level x is given by

$$I_x(t) = u(x - X(t)).$$

$I_x(t) = F_{X(t)}(x)$ 1st-order CDF of $X(t)$.

  - It is a CT random process taking only 0 and 1 on $t \in (-\infty, \infty)$.
  - By choosing a different value of x, we can construct a different indicator process.

- Def. A random process is ergodic in the distribution function if

$$\mathbf{A}[I_x(t)] = F_{X(t)}(x) = E[I_x(t)]$$

for all continuous points x.

  - MS ergodic if convergence in the MS sense
  - Lemma. $I_x(t)$ is WSS if $X(t)$ is strictly stationary to order 2.
    - Proof. Since $\mathbf{E}[I_x(t)] = F_{X(t)}(x)$ and $\mathbf{E}[I_x(t+\tau)I_x(t)] = F_{X(t+\tau),X(t)}(x,x)$, we have the conclusion.
  - Lemma. When $X(t)$ is strictly stationary to order 2, it is MS ergodic in the distribution function iff

$$\frac{1}{2T}\int_{-2T}^{2T} \left(1 - \frac{|\tau|}{2T}\right) C_{I_x I_x}(\tau) d\tau \to 0 \text{ as } T \to \infty,$$

A real RV X,

$F_X(x) = E[1_{\{X \le x\}}]$

$= P(X \le x)$

$= 1 \times P(X \le x) + 0 \times P(X > x)$

$= P(X \le x)$

$= F_X(x)$

$\overset{!}{=} E[u(x - X)]$

$= 1 \times P(x - X \ge 0) + 0 \times P(x - X < 0)$

for all continuous points x.

* Note that

$$C_{I_x I_x}(\tau) = F_{X(t+\tau),X(t)}(x,x) - \{F_{X(t)}(x)\}^2.$$

* Hence, if $X(t+\tau)$ and $X(t)$ are asymptotically independent as τ tend to infinity, then $X(t)$ is MS ergodic in the distribution function.