CM19-230502Tue

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Announcements

1. Quiz #2 will be taken on next Tuesday during the class meeting.
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
   Passwords are ...
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
4. HWs will be assigned. Check the PLMS.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions

7. Sequence of Random Variables
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process

10. Random Walks and Other Applications
11. Spectral Representation
12. Spectrum Estimation
13. Mean-Square Estimation

Review

1. autocorrelation function of a complex-valued random process
   a. pseudo-autocorrelation function or complementary autocorrelation function
2. X(t) = Rcos(2pi f_ct +Theta)
   a. a usual counter-example
   b. If ...
      i. SSS Gaussian
      ii. barX = 0,
      iii. R_XX(tau) = (1/2)cos(2pi f_c tau)
      iv. periodic but mean ergodic
3. autocorrelation function of a finite-power WS sationary real-valued random process X(t)
   a. not an arbitray shape
      i. Lemma 1. symmetry
      ii. Lemma 2. bounded by R_XX(0) where equality holds if ?
      iii. Lemma 3. If zero-mean ergodic without periodic compo, then R_XX ->0 as ...
      iv. Lemma 4. If ergodic without periodic compo and N(t) has R_NN -> 0, then R_XX -> ...
   b. If modulation, then ...

Preview

1. crosscorrelation functions
2. measurement of correlation functions

6.3 CORRELATION FUNCTION

- the autocorrelation function
- the cross-correlation function
- other correlation-type functions
- their properties

Autocorrelation Function and Its Properties

- Throughout this section, we almost always assume $X(t)$ and $X[n]$ are real-valued random processes. However, definitions will be given generally for complex-valued processes.
- Def. The autocorrelation function of a CT random process $X(t)$ is defined by

$$R_{XX}(t_1, t_2) = \mathbf{E}[X(t_1)X^*(t_2)].$$

  – It is often written by letting $t_1 = t + \tau$ and $t_2 = t$ with τ a real number as

$$R_{XX}(t + \tau, t) = \mathbf{E}[X(t + \tau)X^*(t)]$$

  where t is time and τ is offset.

$f : \mathbb{C} \to \mathbb{C}$ $\quad f^* : \mathbb{C} \to \mathbb{C}$

$f(z) = \cos z$ $\quad f^*(z) \triangleq (f(z))^*$

$\cancel{f(z)^*} \leftarrow$

$f^*(z) \leftarrow$

- Def. A CT random process is called a second-order process, a regular process, or a finite-power process if

$$\mathbf{E}[|X(t)|^2] < \infty, \forall t,$$

  – i.e., $R_{XX}(t, t) < \infty, \forall t$

- Def. A CT random process is called white if there exists $q(t)$ such that

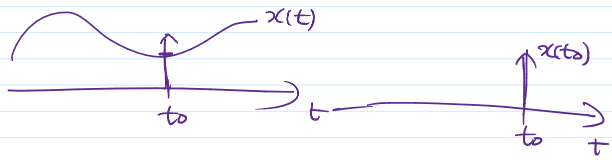
Payments

$$R_{XX}(t_1, t_2) = \boxed{q(t_1)}\delta(t_1 - t_2), \forall (t_1, t_2) \in \mathbb{R}^2.$$

  – Why not $q(t_1, t_2)$? It is because $q(t_1, t_2)\delta(t_1 - t_2) = q(t_1, t_1)\delta(t_1 - t_2)$.

$$R_X(t_1, t_2) = f(t_1, t_2)\delta(t_1 - t_2)$$
$$= f(t_1, t_1)\delta(t_1 - t_2)$$
$$= f(t_2)\delta(t_1 - t_2)$$

$$x(t)\,\delta(t - t_0) = x(t_0)\,\delta(t - t_0)$$



- Def. A CT random process is WSS if there exist c and $R_{XX}(\tau)$ such that

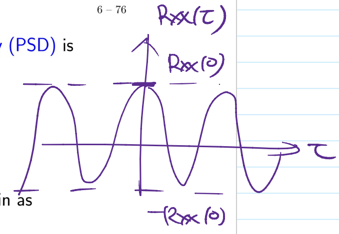
$$\mu_X(t) \triangleq \mathbf{E}[X(t)] = c, \forall t$$

  and

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2), \forall (t_1, t_2) \in \mathbb{R}^2.$$

  – Example. Consider $X(t) = a\cos(2\pi f_0 t + \Theta)$, where $f_0 \neq 0$ is a constant and Θ is a uniform random variable on $[0, 2\pi)$.
    * $X(t)$ is WSS because $\mu_X(t) = 0, \forall t$ and $R_{XX}(t + \tau, t) = (a^2/2)\cos(2\pi f_0 \tau)$.
    * $R_{XX}(\tau)$ has a periodic component, (but satisfies the necessary and sufficient condition to be mean-ergodic in the MS sense.)

- Theorem. $R_{XX}(\tau)$ of a real-valued at least WSS CT random process $X(t)$ cannot have an arbitrary shape.



– This fact will be more apparent when the power spectral density (PSD) is introduced in the next chapter.

- Lemmas. If $X(t)$ is a real-valued at least WSS CT random process
  – Lemma 1. ..., then $R_{XX}(\tau)$ has even symmetry, i.e.,

$$R_{XX}(-\tau) = R_{XX}(\tau).$$

  – Lemma 2. ...., then $R_{XX}(\tau)$ is bounded by its value at the origin as

$$|R_{XX}(\tau)| \le R_{XX}(0),$$

where the upper bound $R_{XX}(0) = \mathbf{E}[X^2(t)]$ is equal to the mean-squared value called the power in the process.

  – Lemma 3. ... and $X(t)$ has a periodic component, then $R_{XX}(\tau)$ will have a periodic component with the same period.
  – Lemma 4. ..., ergodic, $\mathbf{E}[X(t)] = 0, \forall t$, $X(t)$ has no periodic component, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = 0.$$

Recall the above example having a periodic component.

  – Lemma 5. ..., ergodic, $\mathbf{E}[X(t)] = \bar{X} \neq 0, \forall t$, $X(t)$ has no periodic component, ergodic, and $N(t) \triangleq X(t) - \bar{X}$ has $R_{NN}(\tau) \to 0$ as $|\tau| \to \infty$, then

$$\lim_{|\tau|\to\infty} R_{XX}(\tau) = \bar{X}^2.$$

Example: Given a stationary ergodic process $X(t)$ with no periodic component and the autocorrelation function

$$R_{XX}(\tau) = 25 + \frac{4}{1 + 6\tau^2},$$

find its mean and variance.

(solution)

From Property (4), the mean value is $\mathbf{E}[X(t)] = \bar{X} = \sqrt{25} = \pm 5$.

The variance is

$$\begin{aligned} \sigma_X^2 &= \mathbf{E}[X^2(t)] - (\mathbf{E}[X(t)])^2 \\ &= R_{XX}(0) - \bar{X}^2 \\ &= (25 + 4) - 25 = 4. \end{aligned}$$

Example: Let $X(t)$ be a WSS random process with autocorrelation function

$\bar{X} = 0, \forall t,$ $R_{XX}(\tau) = e^{-a|\tau|}, \quad a > 0$

Determine the autocorrelation function of $Y(t) = X(t)\cos(2\pi f_0 t + \Theta)$. $[0, 2\pi)$

$R_{XX}(\tau)$ τ

(solution) $\mathbf{E}[Y(t)] = \mathbf{E}[X(t)\cos(2\pi f_0 t + \Theta)] = \underbrace{\mathbf{E}[X(t)]}_{0}\ \underbrace{\mathbf{E}[\cos(2\pi f_0 t + \Theta)]}_{0}$

$$\begin{aligned} R_{YY}(t+\tau, t) &= \mathbf{E}[Y(t+\tau)Y(t)] = \cdots \\ &= \frac{1}{2} R_{XX}(\tau)\cos(2\pi f_0 \tau) \\ &= R_{YY}(\tau) \end{aligned}$$

cf) $Z(t) = X(t)\cos(2\pi f_0 t + \Theta)$ $\mathbf{E}[Z(t)] = 0 \ \forall t$

$R_{ZZ}(t+\tau, t) = \mathbf{E}[X(t+\tau)\cos(2\pi f_0(t+\tau) + \Theta)\, X(t)\cos(2\pi f_0 t + \Theta)]$

$= R_{XX}(\tau)\ \cos(2\pi f_0(t+\tau) + \Theta)\cos(2\pi f_0 t + \Theta)$

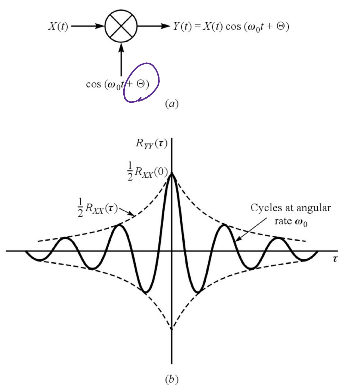


Figure 6.3-1 (a) Amplitude modulation of a carrier with random phase by a random process $X(t)$. (b) A plot of the autocorrelation function of $Y(t)$.

Cross-Correlation Function and Its Properties

- Def. The cross-correlation function between two CT random processes $X(t)$ and $Y(t)$ is defined by

$$R_{XY}(t+\tau, t) = \mathbf{E}[X(t+\tau)Y^*(t)].$$

- Def. Two real-valued CT random processes $X(t)$ and $Y(t)$ are called uncorrelated, if $R_{XY}(t_1,t_2) = \mathbf{E}[X(t_1)]\mathbf{E}[Y^*(t_2)], \forall(t_1,t_2) \in \mathbb{R}^2$.
  – If $X(t)$ and $Y(t)$ are statistically independent, then they are uncorrelated.
- Def. Two real-valued CT random processes $X(t)$ and $Y(t)$ are called orthogonal, if $R_{XY}(t_1,t_2) = 0, \forall(t_1,t_2) \in \mathbb{R}^2$.
- Def. Two WSS CT random processes $X(t)$ and $Y(t)$ are called jointly WSS, if there exists a single-variable function $R_{XY}(\tau)$ such that

$$R_{XY}(t+\tau,t) = R_{XY}(\tau), \forall(t,\tau) \in \mathbb{R}^2,$$

i.e., $R_{XY}(t+\tau,t)$ is independent of absolute time.
  – If $X(t)$ and $Y(t)$ are statistically independent and WSS

$$R_{XY}(\tau) = \bar{X}\bar{Y}.$$

$$R_{XY}(t+\tau,t) = E[X(t+\tau)Y^*(t)] = E[X(t+\tau)]E[Y^*(t)] = \bar{X}\;\bar{Y}^*$$

$Y(t)=X(t)$

- Properties of the cross-correlation function of at least jointly WSS real-valued random processes

(1)	$R_{XY}(-\tau) = R_{YX}(\tau)$	(6.3-1)
(2)	$\lvert R_{XY}(\tau)\rvert \le \sqrt{R_{XX}(0)R_{YY}(0)}$	(6.3-2)
(3)	$\lvert R_{XY}(\tau)\rvert \le \frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$	(6.3-3)

$R_{XX}(-\tau) = R_{XX}(\tau)$,

$|R_{XX}(\tau)| \le \sqrt{R_{XX}(0)R_{XX}(0)} = R_{XX}(0)$

Remarks

(1) It describes the odd symmetry of $R_{XY}(\tau)$ from definition.

$R_{XY}(t+\tau,t) = E[X(t+\tau)Y(t)] = R_{XY}(\tau)$

$R_{XY}(-\tau) = E[X(t-\tau)Y(t)]$
$= E[Y(t)X(t-\tau)]$
$= R_{YX}(\tau), \quad \forall\tau$

(2) It can be proven by expanding the inequality $E(\alpha X(t+\tau)+Y(t))^2 \ge 0$.

$$\mathbf{E}[\{\cancel{Y(t+\tau)+\alpha X(t)}\}^2] \ge 0, \forall\alpha \in \mathbb{R}$$

and using the discriminant D.

$E\{(\ )(\ )\}$
$= \alpha^2 R_{XX}(0) + 2\alpha R_{XY}(\tau) + R_{YY}(0) \ge 0, \ \forall\alpha\in\mathbb{R}$

$D = (2R_{XY}(\tau))^2 - 4R_{XX}(0)R_{YY}(0) \le 0.$

(3) The bound in (2) is tighter than the bound in (3), because the geometric mean of two positive real numbers cannot exceed their arithmetic mean, that is,

$$\sqrt{R_{XX}(0)R_{YY}(0)} \le \frac{1}{2}[R_{XX}(0)+R_{YY}(0)]$$

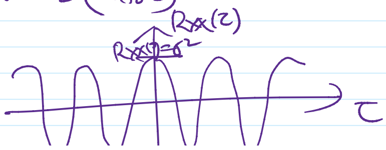
Example: Let two random processes $X(t)$ and $Y(t)$ be defined by

$$X(t) = A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)$$
$$Y(t) = B\cos(2\pi f_0 t) - A\sin(2\pi f_0 t)$$

$\bar{X}=0$, $\bar{Y}=0$

$R_{XX}(t+\tau,t)$

$= E\{[A\cos(2\pi f_0(t+\tau)) + B\sin(2\pi f_0(t+\tau))][A\cos(2\pi f_0 t) + B\sin(2\pi f_0 t)]\}$
$= \sigma^2 \cos(2\pi f_0(t+\tau))\cos(2\pi f_0 t) + \sin(2\pi f_0(t+\tau))\sin(2\pi f_0 t)$
$= \sigma^2 \cos(2\pi f_0(t+\tau) - 2\pi f_0 t)$
$= \sigma^2 \cos(2\pi f_0 \tau)$



where A and B are RVs and f_0 is a non-zero constant. Show that $X(t)$ and $Y(t)$ are jointly WSS if A and B are uncorrelated, zero-mean RVs with the same variance σ^2.

(solution) Check that $X(t)$ and $Y(t)$ are WSS.

The cross-correlation function of $X(t)$ and $Y(t)$ can be computed as

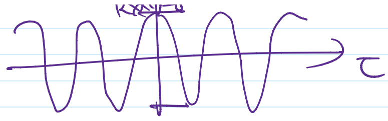
$$R_{XY}(t+\tau,t) = -\sigma^2\sin(\omega_0\tau).$$

$X(t)$ and $Y(t)$ are jointly WSS, since $R_{XY}(t+\tau,t)$ depends only on τ.

cf) $X(t) = \mathrm{Re}\{(A-jB)e^{j2\pi f_0 t}\}$
$Y(t) = \mathrm{Re}\{(B+jA)e^{j2\pi f_0 t}\}$

$E(A-jB) = E[B+jA] = 0$
$E[|A-jB|^2] = E[|B+jA|^2] = E[A^2]+E[B^2] = 2\sigma^2$
$E[(A-jB)^2] = E[A^2-2jAB-B^2] = 0$

$$E[|A-jB|^2] = E[|B+jA|^2] = E[A^2]+E[B^2] = 2\sigma^2$$
$$E[(A-jB)^2] = E[A^2-2jAB-B^2] = 0$$



$R_{YY}(\tau) = R_{XX}(\tau), \ \forall \tau.$

Covariance Functions

- The autocovariance function of two real-valued random processes $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{XX}(t+\tau,t) &\triangleq \mathbf{E}[\{X(t+\tau)-\mathbf{E}[X(t+\tau)]\}\{X(t)-\mathbf{E}[X(t)]\}^*] \\ &= R_{XX}(t+\tau,t)-\mathbf{E}[X(t+\tau)]\mathbf{E}[X(t)] \end{aligned}$$

- The cross-covariance function of two real-valued random processes $X(t)$ and $Y(t)$ is defined by

$$\begin{aligned} C_{XY}(t+\tau,t) &\triangleq \mathbf{E}[\{X(t+\tau)-\mathbf{E}[X(t+\tau)]\}\{Y(t)-\mathbf{E}[Y(t)]\}^*] \\ &= R_{XY}(t+\tau,t)-\mathbf{E}[X(t+\tau)]\mathbf{E}[Y^*(t)] \end{aligned}$$

- For real random processes that are at least jointly WSS

$$C_{XX}(\tau) = R_{XX}(\tau)-\bar{X}^2 \quad \text{and} \quad C_{XY}(\tau) = R_{XY}(\tau)-\bar{X}\bar{Y}$$

- The variance of a random process $X(t)$ is defined by $C_{XX}(t,t)$.

  For a real WSS random process,

$$\sigma_X^2 = \mathbf{E}[\{X(t)-\mathbf{E}[X(t)]\}^2] = R_{XX}(0)-\bar{X}^2$$

- Two real random processes are called uncorrelated if

$$C_{XY}(t+\tau,t) = 0$$

or equivalently,

$$R_{XY}(t+\tau,t) = \mathbf{E}[X(t+\tau)]\mathbf{E}[Y(t)]$$

- Independence implies uncorrelatedness. The converse is not true in general, although it is true for joint Gaussian process.

Discrete-Time Processes and Sequences

- For DT processes $X(nT_s)$ and $Y(nT_s)$,

  the mean function: Mean $\triangleq \mathbf{E}[X(nT_s)]$
  - the autocorrelation function: $R_{XX}(nT_s, nT_s + kT_s) = \mathbf{E}[X(nT_s)X(nT_s + kT_s)]$
  - the autocovariance function: $C_{XX}(nT_s, nT_s + kT_s)$
  - the cross-correlation function: $R_{XY}(nT_s, nT_s + kT_s) = \mathbf{E}[X(nT_s)Y(nT_s + kT_s)]$
  - the cross-covariance function: $C_{XY}(nT_s, nT_s + kT_s)$
- For random processes that are at least jointly WSS,

$$\mathbf{E}[X(nT_s)] = \bar{X} \qquad \mathbf{E}[Y(nT_s)] = \bar{Y}$$

$$R_{XX}(nT_s, nT_s + kT_s) = R_{XX}(kT_s) \tag{6.3-4}$$
$$R_{YY}(nT_s, nT_s + kT_s) = R_{YY}(kT_s) \tag{6.3-5}$$
$$C_{XX}(nT_s, nT_s + kT_s) = R_{XX}(kT_s) - \bar{X}^2 \tag{6.3-6}$$
$$C_{YY}(nT_s, nT_s + kT_s) = R_{YY}(kT_s) - \bar{Y}^2 \tag{6.3-7}$$
$$R_{XY}(nT_s, nT_s + kT_s) = R_{XY}(kT_s) \tag{6.3-8}$$
$$R_{YX}(nT_s, nT_s + kT_s) = R_{YX}(kT_s) \tag{6.3-9}$$
$$C_{XY}(nT_s, nT_s + kT_s) = R_{XY}(kT_s) - \bar{X}\bar{Y} \tag{6.3-10}$$
$$C_{YX}(nT_s, nT_s + kT_s) = R_{YX}(kT_s) - \bar{Y}\bar{X} \tag{6.3-11}$$

6.4 MEASUREMENT OF CORRELATION FUNCTIONS

- In the real world, we can never know the true correlation functions of two random processes.

  This is because we never have all sample functions of the ensemble.
- Only a portion of one sample function from each process is available for measurements.

  Need to determine time averages based on finite time portions of single sample functions, taken large enough to approximate true results for ergodic processes
- Measure the approximate time cross-correlation function of two jointly ergodic random processes

$\frac{1}{2T}\int_{t_1}^{t_1+2T} x(t-T)\, y(t-T-\tau)\, dt$

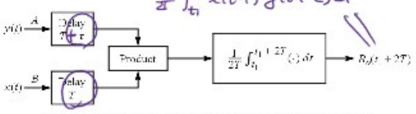
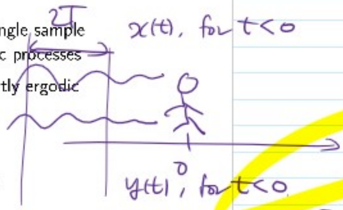


Figure 6.4-1 A time cross-correlation function measurement system



- The output is

$$R_o(t_1 + 2T) = \frac{1}{2T}\int_{t_1 - T}^{t_1 + T} x(t)y(t-\tau)\,dt$$

$R_{XY}(\tau)$

You want to estimate

If X & Y are jointly-ergodic then

$A[X(t+\tau)Y(t)] = E[X(t+\tau)Y(t)]$

$x(t)y$

$\simeq \frac{1}{2T}\int_{2T} x(t+\tau)y(t)\,dt$

$\simeq \frac{1}{2T}\int_{2T} x(t)y(t-\tau)\,dt$

If we choose $t_1 = 0$ and assume T is large,

$$R_o(2T) = \frac{1}{2T}\int_{-T}^{T} x(t)y(t-\tau)\,dt \approx \mathcal{R}_{xy}(\tau) = R_{XY}(\tau)$$

- τ is varied to obtain the complete function

>> Example 6.4-1.

6.5 SUMMARY

- random process
- continuous-time and discrete-time processes
- deterministic and nondeterministic random processes
- stationarity
- independence
- Gaussian and Poisson processes
- complex random processes
- ergodicity
- autocorrelation, cross-correlation, autocovariance, cross-covariance functions
- measuring a correlation function

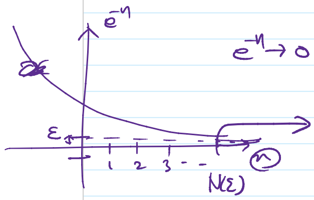
7.4 Stochastic Convergence and Limit Theorem

Equality vs. Approximate Equality of Two Real Numbers

- Equality of Two Real Numbers
  - Given two real numbers x and y, we have $x = y$ xor $x \neq y$.
  - Lemma. Two real numbers x and y, we have $x = y$ if and only if $x - y = 0$.
    - ... if and only if $x - y \leq 0$ and $x - y \geq 0$.
    - ... if and only if $|x - y| \leq 0$.
  - Proposition. Given two real numbers x and y, we have $x = y$ if and only if

$$|x - y| \leq \epsilon, \forall \epsilon > 0.$$ iff $|x-y| < \varepsilon, \forall \varepsilon > 0$

    - Proof by contraposition. $x \neq y$ iff $\exists \epsilon > 0, |x - y| > \epsilon$.
    - (o) ... if and only if $|x - y| < \epsilon, \forall \epsilon > 0$.
    - (x) ... if and only if $|x - y| < \epsilon, \forall \epsilon \geq 0$.
- Approximate Equality of Two Real Numbers
  - What about $x \approx y$ or $x - y \approx 0$?
  - Mathematical tools to handle 'approximately equal' or 'almost equal' are
    - upper and lower bounding and



$e^{-n} \to 0$ as $n \to \infty$

  * convergence of a sequence of real numbers

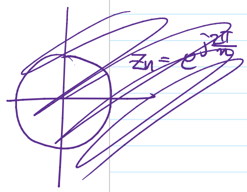
- Given a sequence $(x_n)_{n\geq 1}$ of real numbers and y, we may have the situation where
  - $x_n \neq y, \forall n$ but
  - we want to claim that $x_n \approx y$ for sufficiently large n.
    * Example. $x_n = e^{-n}$, $y = 0 \Longrightarrow x_n \neq y, \forall n$, but $x_n \approx y$ for sufficiently large n..
    * The notion of the convergence of a sequence of real numbers is to handle such an approximate equality mathematically precisely.
    * First, we need to review the convergence of a sequence of real numbers.

- Def. A sequence $(x_n)_{n\geq 1}$ of real numbers converges to a real number x if

$$\forall \varepsilon > 0, \exists N(\varepsilon) \overset{\in\mathbb{N}}{\cancel{>}} 0, \text{ s.t. } n \geq N(\varepsilon) \Longrightarrow |x_n - x| \leq \varepsilon.$$

  - "For every epsilon positive,
    there exists N of epsilon positive such that
    the difference between x_n and x is less than epsilon
    for all n greater than or equal to N of epsilon."
  - x is called the limit of the sequence.
  - The definition can serve to check the convergence of a sequence only when the candidate limit is provided. $\to$ the Cauchy sequence in $\mathbb{R}$

  - Notations
    * $\lim_{n\to\infty} x_n = x$
    * $x_n \to x$ as $n \to \infty$ (as n tends to infinity)
    * $x_n \to x$ (x_n converges to x)
    * $\forall \varepsilon > 0, |x_n - x| \leq \varepsilon$ for sufficiently large n
    * $x_n \approx x$ for sufficiently large n
  - Properties
    * translation invariance: $x_n \to x$ iff $(x_n - x) \to 0$
    * linearity: $x_n \to x$ and $y_n \to y \Longrightarrow ax_n + by_n \to ax + by$
      · Consider $y_n = 1, \forall n$. Then, $x_n \to x \Longrightarrow ax_n + b \to ax + b$.



- Def. A sequence $(x_n)_n$ (not necessarily of real numbers) is called a Cauchy sequence if

$$\forall \varepsilon > 0, \exists N(\varepsilon) > 0, \text{ s.t. } m,n \geq N(\varepsilon) \Longrightarrow |x_n - x_m| \leq \varepsilon.$$

  - Lemma. For a sequence $(x_n)_n$ of real numbers, it is Cauchy if and only if it converges.
    [Leon-Garcia] This lemma says that "the maximum variation in the sequence for points beyond $N(\varepsilon)$ is less than ε."