CM20-230504Thu

2023년 2월 23일 목요일 10:44

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Announcements

1. Quiz #2 will be taken on next Tuesday during the class meeting.
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
4. HWs will be assigned. Check the PLMS.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

9. General Concepts
   1. RP concept
   (2) Stationarity and Independence
   (5) Gaussian RPs
   (6) Poisson RPs
   (7) Complex RPs
   (2) Time average and Ergodicity
   (3) Correlation functions
   (4) Measurement of correlation functions
   9.1
   - positive semi-definiteness of a correlation function
   - alpha-dependent random process
   - correlation alpha-dependent random process
   - white noise in terms of correlation function
   - mean-square periodic random process
   - cyclostationary random process
   7. Sequence of Random Variables
   10. Random Walks and Other Applications
   11. Spectral Representation
   12. Spectrum Estimation
   13. Mean-Square Estimation

Review

1. definitions of correlation and covariance functions
2. measurement of correlation functions
   - ergodic assumption
3. Equality of two real or complex numbers

$x=y \Leftrightarrow \forall \varepsilon>0,\ |x-y|\le\varepsilon$ (O) $\Leftrightarrow \forall \varepsilon>0,\ |x-y|\le\varepsilon$ (O)

Caution! $\forall \varepsilon\ge 0,\ |x-y|\le\varepsilon$ (Δ) $\forall \varepsilon\ge 0,\ |x-y|<\varepsilon$ (X) Avoid using it

4) $p\Leftrightarrow q$
$\equiv (p\Rightarrow q)\wedge(q\Rightarrow p)$
$\equiv (p\Rightarrow q)\wedge(\neg p\Rightarrow\neg q)$
$0<0$ (False)

4. approximate equality
   - Bounding
   - Construct a sequence and show it converges.

$x\simeq y$

Construct $(x_n)_n$ & show $x_n\simeq y$ f.s.l. n

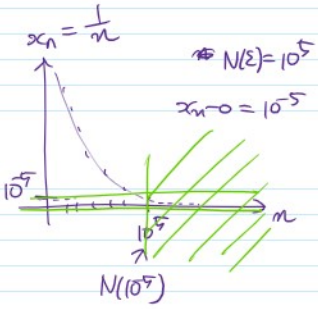
Def. $x_n\to y$ as $n\to\infty$ if

$\forall\varepsilon>0,\ \exists N(\varepsilon)\in\mathbb{N}$ s.t. $|x_n-y|\le\varepsilon$ for $n\ge N(\varepsilon)$ (O) ← Best to use

" " s.t. $|x_n-y|<\varepsilon$ for $n\ge N(\varepsilon)$ (O) ← Ok to use.

Caution $\forall\varepsilon\ge 0,\ \exists N(\varepsilon)\in\mathbb{N}$ s.t. $|x_n-y|\le\varepsilon$ for $n\ge N(\varepsilon)$ (X) ← False

$\frac{1}{n}\to 0$ but $|x_n y|\neq 0, \forall n.$ " " s.t. $|x_n-y|<\varepsilon$ for $n\ge N(\varepsilon)$ (X) ← False



Preview

1. Topics related with autocorrelation functions
2. Back to Chapter 7 of Papoulis

Chapter 9
GENERAL CONCEPTS

Prepared by Jun Ho Cho
based on A. Papoulis & S. U. Pillai, "Probability, Random Variables and Stochastic Processes, 4th ed."

EECE 574, Spring 2021

9.1 Definitions
9.2 Systems with Stochastic Inputs
9.3 The Power Spectrum
9.4 Discrete-Time Processes
9.5 APPENDIX Continuity, Differentiation, Integration
9.6 APPENDIX Shift Operators and Stationary Processes

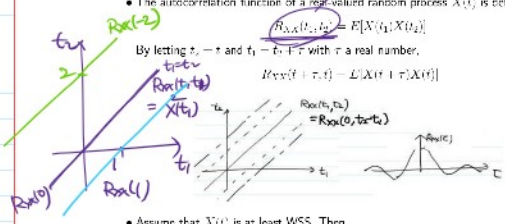
Properties of Autocorrelation Function

- The autocorrelation function of a real-valued random process $X(t)$ is defined by

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] \tag{9.1-5}$$

By letting $t_1 = t$ and $t_2 = t + \tau$ with τ a real number,

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] \tag{9.1-26}$$



- Assume that $X(t)$ is at least WSS. Then

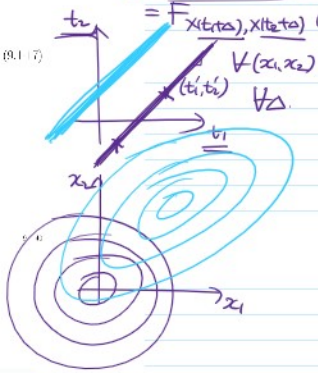
$$R_{XX}(\tau) = E[X(t-\tau)X(t)] \tag{9.1-7}$$

(1) $R_{XX}(\tau)$ is bounded by its value at the origin:

$$R_{XX}(\tau) \le R_{XX}(0)$$

Recall
Def. $X(t)$ is stationary to order 2 if

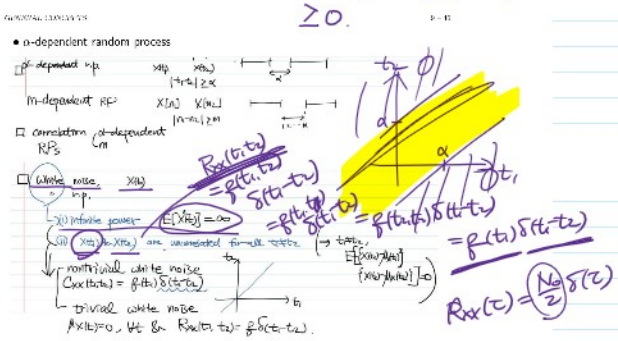
$$F_{X(t_1)X(t_2)}(x_1,x_2) = F_{X(t_1+\tau)X(t_2+\tau)}(x_1,x_2) \quad \forall (x_1,x_2), \forall (t_1,t_2), \forall \Delta.$$

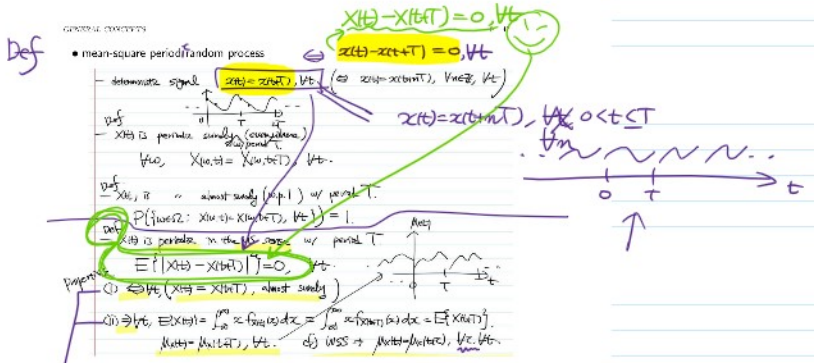


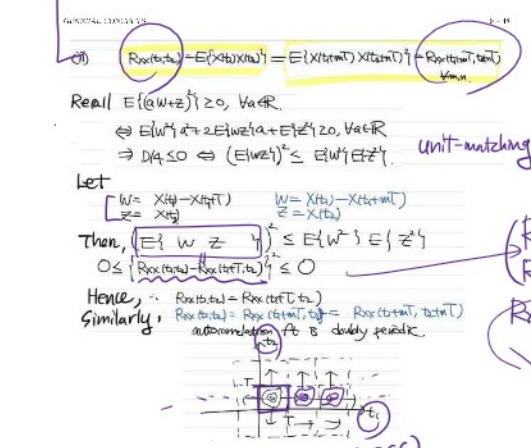
□ Back to $|R_{XX}(\tau)| \le R_{XX}(0)$ (WSS $X(t)$) — cross power

$E[X(t+\tau)X(t)]$ $\quad E[X(t-\tau)X(t+\tau)]$
$\quad\quad\quad

≥ 0.





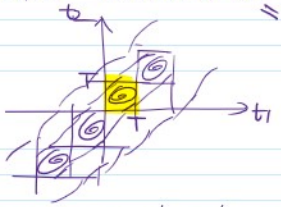


$(R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2)$
$R_{XX}(t_1,t_2) = R_{XX}(t_1, t_2+T)$
$R_{XX}(t_1,t_2) = R_{XX}(t_1+mT, t_2+nT)$
$\forall (m,n) \in \mathbb{Z}^2$
$\forall (t_1,t_2) \in \mathbb{R}^2$

Def. A CT RP is cyclostationary w/ cycle period T if

(i) $\mu_X(t) = \mu_X(t+T)$, $\forall t$

(ii) $R_{XX}(t_1,t_2) = R_{XX}(t_1+T, t_2+T)$, $\forall (t_1,t_2)$
$= R_{XX}(t_1+mT, t_2+mT)$



Lemma. If $X(t)$ is MS periodic w/ period T then it is cyclostationary w/ cycle period T.

7.4 Stochastic Convergence and Limit Theorem

Equality vs. Approximate Equality of Two Real Numbers

- Equality of Two Real Numbers

  Given two real numbers x and y, we have $x - y$ xor $x \neq y$.

  – Lemma. Two real numbers x and y, we have $x = y$ if and only if $x - y = 0$.

  * ... if and only if $x - y \leq 0$ and $x - y \geq 0$.
  * ... if and only if $|x - y| \leq 0$.

  – Proposition. Given two real numbers x and y, we have $x = y$ if and only if

  $$|x - y| \leq \epsilon, \forall \epsilon > 0.$$

  iff $|x-y| < \varepsilon, \forall \varepsilon > 0$

  * Proof by contraposition. $x \neq y$ iff $\exists \epsilon > 0,\ |x - y| > \epsilon$.
  * (o) ... if and only if $|x - y| < \epsilon, \forall \epsilon > 0$.
  * (x) ... if and only if $|x - y| < \epsilon, \forall \epsilon \geq 0$. (x)

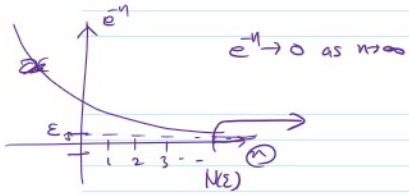
  $|x-y| \leq \varepsilon \ \forall \varepsilon \geq 0$ (o)

- Approximate Equality of Two Real Numbers

  – What about $x \approx y$ or $x - y \approx 0$?

  Mathematical tools to handle 'approximately equal' or 'almost equal' are

  * upper and lower bounding and



  * convergence of a sequence of real numbers

- Given a sequence $(x_n)_{n \geq 1}$ of real numbers and y, we may have the situation where

  – $x_n \neq y, \forall n$ but

  we want to claim that $x_n \approx y$ for sufficiently large n.

  * Example. $x_n = e^{-n}, y = 0 \Rightarrow x_n \neq y, \forall n$, but $x_n \approx y$ for sufficiently large n..
  * The notion of the convergence of a sequence of real numbers is to handle such an approximate equality mathematically precisely.
  * First, we need to review the convergence of a sequence of real numbers.

- Def. A sequence $(x_n)_{n \geq 1}$ of real numbers converges to a real number x if

  $$\forall \varepsilon > 0, \exists N(\varepsilon) > 0, \text{ s.t. } n \geq N(\varepsilon) \longrightarrow |x_n - x| \leq \varepsilon.$$

  "For every epsilon positive,
  there exists N of epsilon positive such that
  the difference between x_n and x is less than epsilon
  for all n greater than or equal to N of epsilon."
  x is called the limit of the sequence.

  – The definition can serve to check the convergence of a sequence only when the candidate limit is provided. → the Cauchy sequence in $\mathbb{R}$

$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ s.t.
$|x_n - x| \leq \varepsilon$, for $n \geq N(\varepsilon)$

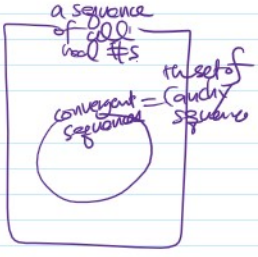
  – Notations

  * $\lim_{n \to \infty} x_n = x$
  * $x_n \to x$ as $n \to \infty$ (as n tends to infinity)
  * $x_n \to x$ (x_n converges to x)
  * $\forall \varepsilon > 0, |x_n - x| \leq \varepsilon$ for sufficiently large n
  * $x_n \approx x$ for sufficiently large n

  – Properties

  * translation invariance: $x_n \to x$ iff $(x_n - x) \to 0$
  * linearity: $x_n \to x$ and $y_n \to y \longrightarrow ax_n + by_n \to ax + by$
    · Consider $y_n = 1, \forall n$. Then, $x_n \to x \longrightarrow ax_n - b \to ax + b$.

$\sqrt{2} = 1.414\cdots$
$x_1 = 1.4$
$x_2 = 1.41$
$x_3 = 1.414\cdots$



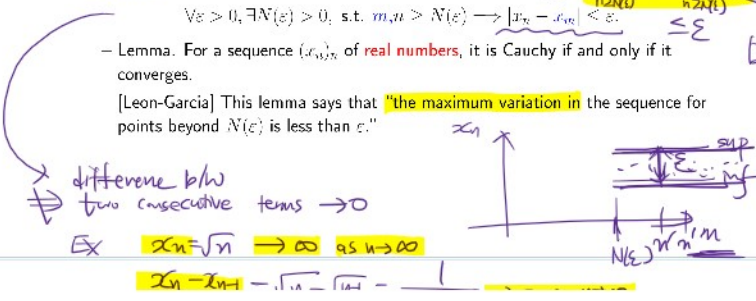
- Def. A sequence $(x_n)_n$ (not necessarily of real numbers) is called a Cauchy sequence if

  $$\forall \varepsilon > 0, \exists N(\varepsilon) > 0, \text{ s.t. } m, n > N(\varepsilon) \longrightarrow |x_n - x_m| < \varepsilon.$$

  $\Rightarrow \sup_{n \geq N(\varepsilon)} x_n - \inf_{n \geq N(\varepsilon)} x_n \leq \varepsilon$

  – Lemma. For a sequence $(x_n)_n$ of real numbers, it is Cauchy if and only if it converges.

  [Leon-Garcia] This lemma says that "the maximum variation in the sequence for points beyond $N(\varepsilon)$ is less than ε."



Def. a metric space (distance-)

$x_n, x_{n'}$

관련된 definition

- a metric space: a set of elements and an accompanying distance metric
  - A distance metric: two variable function into a non-negative real such that
    - 1. homogeneity
    - 2. commutativity
    - 3. triangle inequality

a complete metric space (one where all Cauchy sequences converge to a

difference b/w two consecutive terms $\to 0$

Ex $x_n = \sqrt{n} \longrightarrow \infty$ as $n \to \infty$

$x_n - x_{n+1} = \sqrt{n} - \sqrt{n+1} = \frac{-1}{\sqrt{n} + \sqrt{n+1}} \to 0$

⇒ two consecutive terms $\to 0$

Ex $x_n = \sqrt{n} \to \infty$ as $n \to \infty$

$x_n - x_{n-1} = \sqrt{n} - \sqrt{n-1} = \frac{1}{\sqrt{n} + \sqrt{n-1}} \to 0$ as $n \to \infty$

- 1. homogeneity
- 2. commutativity
- 3. triangle inequality

a complete metric space (one where all Cauchy sequences converge to a limit)