CM22-230511Thu

Announcements

1. Quiz #2 wa taken. However, one student hasn't yet taken it. No discussion on the quiz, please.
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
   Passwords are ...
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
4. HWs will be assigned. Check the PLMS.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

- Pa 9.1 Definitions
- Pe 6 Random Process--Temporal Characteristics
- Pa 7.4 Stochastic Convergence and Limit Theorems
- Pa 9.2 Systems with Stochastic Inputs
- Pa 9.3 The Power Spectrum
- Pe 7 Random Process--Spectral Characteristics
- Pe 8 Linear Systems with Random Inputs
- Pe 9 Optimal Linear Systems

Review

1. two real- or complex-valued functions
   a. Equality
      i. equal everywhere
      ii. equal almost everywhere

Preview

1. two real- or complex-valued functions
   a. Equality
   b. Approximate Equality
      - Construct a sequence and show it converges.
      - 5 modes of convergence
2. two random variables
   a. Equality
   b. Approximate Equality
      - Construct a sequence and show it converges.
      - 4 modes of convergence

$X : \Omega_1 \to \boxed{R_1}$

$Y : \Omega_2 \to \boxed{R_2}$

(i) $\Omega_1 = \Omega_2 = \boxed{\Omega}$ an uncountable set $[0,1]$

(ii) $R_1, R_2 \subset \mathbb{R}$ $\quad \mathbb{R}.$

$\mathbb{C}$

☐ Def. $\boxed{X:\Omega\to\mathbb{R},\ Y:\Omega\to\mathbb{R}}$

$\boxed{X=Y}$ if $\forall s\in\Omega,\ X(s)=Y(s).$

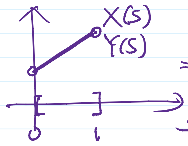
Notation. $X=Y$ everywhere (on Ω)

$X=Y$ (e) e. $\qquad X \overset{e}{=} Y$

$\equiv$ if $\{s\in\Omega : X(s)=Y(s)\}=\Omega$

$\equiv$ if $\{s\in\Omega : X(s)\neq Y(s)\}=\emptyset$ \emptyset

↑ the empty set



$\Omega=[0,1]$

$\{s\in\Omega : X(s)=Y(s)\}$

$=(0,1)\neq\Omega$

$\subset\Omega$

$m\{\quad\}=1$

$m(\Omega)=1$

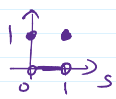
☐ Def. $X:\Omega\to\mathbb{R},\ Y:\Omega\to\mathbb{R}$ (on Ω)

$X=Y$ almost everywhere if ... $\forall\varepsilon>0,\ |X(s)-Y(s)|\le\varepsilon$

$m(\{s\in\Omega : \boxed{X(s)=Y(s)}\}) = m(\Omega)$

$\equiv m(\{s\in\Omega : \boxed{X(s)\neq Y(s)}\}) = 0.$ $\quad \equiv m(\{s\in\Omega : \forall\varepsilon>0,\ |X(s)-Y(s)|>\varepsilon\})$

$\{0,1\}$ $\qquad$ a.e.



☐ Lemma. $\boxed{X:\Omega\to\mathbb{R} \text{ and } Y:\Omega\to\mathbb{R} \text{ are a.e.}}$ iff $\int_\Omega |X(s)-Y(s)|^p\, ds = 0$

$p=2$

$\int_0^1 |X(s)-Y(s)|^2 ds$

$=0.$

☐ Lemma. real $p\ge1$

"

$\forall\varepsilon>0,\ m(\{s\in\Omega : |X(s)-Y(s)|>\varepsilon\})=0.$

- Def. (almost everywhere equality) Given two functions $X : \mathcal{D} \to \mathbb{R}$ and $Y : \mathcal{D} \to \mathbb{R}$, they are equal almost everywhere (a.e.) if
$$m(\{s \in \mathcal{D} : X(s) \neq Y(s)\}) = 0, \iff m(\{s \in \mathcal{D} : \forall \varepsilon > 0, |X(s) - Y(s)| > \varepsilon\}) = 0$$
where $m(\cdot)$ is the Borel measure.
  - The Borel measure $m(\cdot)$ gives the size of a real set.
    - $m([0, 1]) = 1$
    - $m([0, 1] \backslash \mathbb{Q}) = 1$

    Thus, $X = Y$ a.e., iff $X(s) \neq Y(s)$ only on a set of measure zero.
  - When $m(\mathcal{D}) < \infty$ a.e. equal iff $m(\mathcal{D}) = m(\{s \in \mathcal{D} : X(s) = Y(s)\})$.
  - Lemma. Two functions $X : \mathcal{D} \to \mathbb{R}$ and $Y : \mathcal{D} \to \mathbb{R}$ are equal almost everywhere (a.e.) iff
$$\forall \varepsilon > 0, m(\{s \in \mathcal{D} : |X(s) - Y(s)| > \varepsilon\}) = 0.$$
    ← many sets parameterized by ε.
    - When $m(\mathcal{D}) < \infty$ a.e. equal iff $\forall \varepsilon > 0, m(\{s \in \mathcal{D} : |X(s) - Y(s)| \leq \varepsilon\}) = m(\mathcal{D})$.
  - Lemma. Two functions $X : \mathcal{D} \to \mathbb{R}$ and $Y : \mathcal{D} \to \mathbb{R}$ are equal almost everywhere (a.e.) iff
$$\int_{\mathcal{D}} |X(s) - Y(s)|^p ds = 0$$
    for some $p \geq 1$, where the integral is the Lebesgue integral.

- Caution! The Def., Lemma, and Proposition have three equivalent expressions. However, the variations of them become no longer equivalent when the convergence of a sequence of functions is considered.
- Approximate Equality of Two Real Functions
  - What about $X(s) \approx Y(s)$?
  - Mathematical tools to handle 'approximately equal' or 'almost equal' are
    - upper and lower bounding and
    - convergence of a sequence of real functions
- Given a sequence $(X_n(s))_n$ of real functions and a real function $X(s)$, we may have the situation where
  - $X_n(s) \neq X(s), \forall n$ but
  - we want to claim that $X_n(s) \approx X(s)$ f.s.l. n.

    Example. Given a square wave $X(s)$, the nth order Fourier series approximation $X_n(s)$ satisfies $X_n(s) \approx X(s)$ f.s.l. n.

- Later in you research, you may need the following notions of distance metric and equivalence class, which are related to the almost everywhere equality.

  Distance Metric
  - We may introduce a metric $d(X(s), Y(s))$ to measure how far away two functions are located.
  - Def. Given a set A, a function $d(x, y)$ of $x, y \in A$ is a distance metric if
    - (non-negativity) $d(x, y) \geq 0$ with equality iff $x = y$ (in some sense),
    - (symmetry) $d(x, y) = d(y, x)$, and
    - (subadditivity–triangle inequality) $d(x, y) + d(y, z) \geq d(x, z)$ hold.
  - Due to the subadditivity requirement, $\int_{-\infty}^{\infty} |X(s) - Y(s)|^p ds$ is not a metric for $0 < p < 1$.
  - "in some sense" may be equal almost everywhere sense.
  - Equivalence Class
    - Given a set A of interest and a distance metric $d : A \times A \to \mathbb{R}_+$,
      - we may partition the set into equivalence classes.
      - A subset that contains all the elements of zero distance to each other is an equivalence class.
      - The equivalence class of an element $x \in A$ is $\{y \in A : d(x, y) = 0\}$.

a metric space (S, d)

$d : S \times S \to \mathbb{R}$

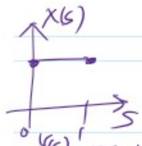
↑ the Cartesian product

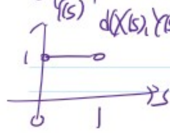
$d(x, y)$

(i) $d(x, y) \geq 0, \forall x, y \in S$ ← equality iff $x = y$

(ii) $d(x, y) = d(y, x) \quad \forall x, y \in S$

(iii) $\forall x, y, z \in S$
$d(x, y) + d(y, z) \geq d(x, z)$





$d(X(s), Y(s)) = \int_0^1 |X(s) - Y(s)|^p ds = 0$

real $p \geq 1$

Def. $X_n(s) \to X(s)$ uniformly

$\forall \varepsilon > 0\ \exists N(\varepsilon) \in \mathbb{N}$ s.t.

$n \geq N(\varepsilon) \Rightarrow |X_n(s) - X(s)| \leq \varepsilon$

Convergence of a Sequence of Functions

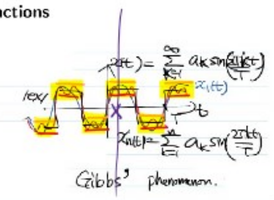
sequence of functions.
$\{x_n(t)\}_{n \geq 1}$

$x(t) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{2\pi k t}{T}\right)$

$x_1(t)$

Convergence of a Sequence of Functions

sequence of functions $\{x_n(t)\}_{n\geq 1}$



$\Downarrow$ — uniform convergence, For s.l.n, $|x_n(t)-x(t)|<\varepsilon, \forall t$
— pointwise convergence, "surely converge"
— almost everywhere ", " converges almost surely"

- Now, we introduce five DIFFERENT modes of convergence all making it mathematically precise that $X_n(s) \approx X(s)$ for sufficiently large n.
  - Though, the mode of convergence is only discussed here, the rate of convergence is also very important.
  - You need to study big O notations and little o notations to describe the rate of convergence.

$\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ s.t.
$\boxed{n \geq N(\varepsilon)} \Rightarrow \boxed{|X_n(s)-X(s)| \leq \varepsilon \text{ for all } s \in \Omega}$

$-\varepsilon \leq X_n(s)-X(s) \leq \varepsilon$

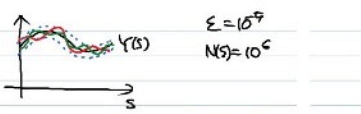
$X(s)-\varepsilon \leq X_n \leq X(s)+\varepsilon$

Lemma. If $X_n \rightrightarrows X$ uniformly on Ω then

$\int_\Omega X(s)\,ds = \lim_{n\to\infty} \int_\Omega X_n(s)\,ds$

1. Def. (uniform convergence) A sequence $(X_n(s))_{n\geq 1}$ of real functions converges uniformly to a real function $X(s)$ on $s \in \mathcal{D}$ if

$$\forall \varepsilon > 0, \exists N(\varepsilon) > 0, \text{ s.t. } n \geq N(\varepsilon) \Longrightarrow |X_n(s) - X(s)| \leq \varepsilon, \forall s \in \mathcal{D}.$$



$\varepsilon = 10^{-9}$
$N(s) = 10^6$

$X_n(s) \rightrightarrows \boxed{Y(s) = \lim_{n\to\infty} X_n(s)}$

$\int_\Omega Y(s)\,ds = \int_\Omega \lim_{n\to\infty} X_n(s)\,ds \neq \lim_{n\to\infty} \int_\Omega X_n(s)\,ds$

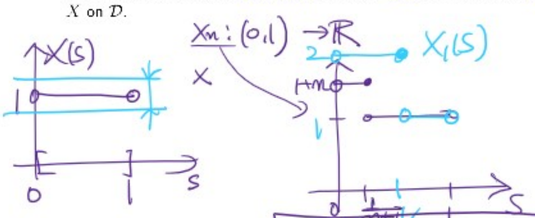
ex/

$\lim_{m\to\infty}\lim_{n\to\infty} \frac{m}{m+n} \neq \lim_{n\to\infty}\lim_{m\to\infty} \frac{m}{m+n}$

2. Def. (everywhere convergence) A sequence $(X_n(s))_{n\geq 1}$ of real functions converges everywhere (e) to a real function $X(s)$ if

$$X_n(s) \to X(s), \forall s \in \mathcal{D}.$$

- $\{s \in \mathcal{D} : X_n(s) \to X(s)\} = \mathcal{D}$.
- $\{s \in \mathcal{D} : X_n(s) \nrightarrow X(s)\} = \emptyset$.
- $\lim_{n\to\infty} X_n(s) = X(s), \forall s \in \mathcal{D}$
- $X_n(s) \to X(s)$ everywhere on $\mathcal{D}$ as $n \to \infty$
- $X_n(s) \to X(s)$ everywhere on $\mathcal{D}$
- pointwise convergence
- Lemma. If X_n converges uniformly to X on $\mathcal{D}$ then X_n converges everywhere to X on $\mathcal{D}$.



U
$\Downarrow$
e
$\Downarrow$
a.e
$\Downarrow$
in measure

in the pth mean $(p\geq 1)$

Def. $X_n \to X$ everywhere if $\forall s \in \Omega$,
$(\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ s.t. $|X_n(s) - X(s)| < \varepsilon)$

Example of an everywhere convergent seq of functions that does not converge uniformly.

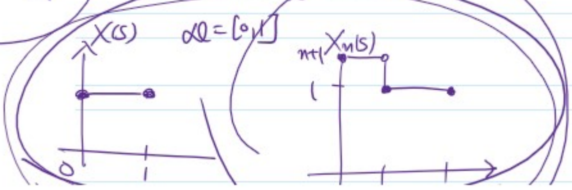
Lemma $X_n \to X$ (e) iff $\{s \in \Omega : X_n(s) \to X(s)\} = \Omega \equiv \{s \in \Omega : X_n(s) \nrightarrow X(s)\} = \phi$

3. Def. (almost everywhere convergence) A sequence $(X_n(s))_{n\geq 1}$ of real functions converges almost everywhere (a.e.) to a real function $X(s)$ on $\mathcal{D}$ if

$$m(\{s \in \mathcal{D} : X_n(s) \nrightarrow X(s)\}) = 0$$

$\cancel{m(\{s\in\Omega : X_n(s) \to X(s)\}) = m(\Omega)}$

- $m(\cdot)$ is the Borel measure.
- When $m(\mathcal{D}) < \infty$, a.e. iff $m(\{s \in \mathcal{D} : X_n(s) \to X(s)\}) = m(\mathcal{D})$.
- $X(s)$ does not converge to $X(s)$ only on a set of measure zero.



- $m(\cdot)$ is the Borel measure.
- When $m(\mathcal{D}) < \infty$, a.e. iff $m(\{s \subset \mathcal{D} : X_n(s) \to X(s)\}) = m(\mathcal{D})$.
- $X(s)$ does not converge to $X(s)$ only on a set of measure zero.
- $\lim_{n\to\infty} X_n(s) = X(s)$ for almost every $s \subset \mathcal{D}$
- $X_n(s) \to X(s)$ a.e. on $\mathcal{D}$ as $n \to \infty$
- $X_n(s) \to X(s)$ a.e. on $\mathcal{D}$
- Lemma. If X_n converges everywhere to X then X_n converges almost everywhere to X.

$\{s \in \Omega : X_n(s) \not\to X(s)\} = \{0\}$

$\Rightarrow m(\{\quad\}) = 0.$

$\int_0^1 |X_n(s) - X(s)|^2 ds = \frac{1}{n+1} n^2 \to \infty.$

cf) X_n, X

$E\{(X_n - X)^2\} \to 0$

cf) $\boxed{X(s) = Y(s) \text{ a.e. iff } \int_\Omega |X(s) - Y(s)|^p ds = 0. \quad p \geq 1}$ $\equiv \forall \varepsilon > 0, \ m(\{s \in \Omega : X(s) - Y(s) \neq 0\}) = 0.$

4. Def. (convergence in the pth mean) A sequence $(X_n(s))_{n\geq 1}$ of real functions converges in the pth mean to a real function $X(s)$ on $\mathcal{D}$ if

$$\int_{s\in\mathcal{D}} |X_n(s) - X(s)|^p ds \to 0$$

for $p \geq 1$.

$\left(\forall \varepsilon > 0, \ \exists N(\varepsilon) \text{ s.t } \left|\int_{s\in D} |X_n(s) - X(s)|^p ds\right| \leq \varepsilon \text{ for } n \geq N(\varepsilon)\right)$

- Note that if we view $\int |X_n(s) - X(s)|^p ds$ as x_n then the convergence in the pth mean is nothing but the convergence of a real sequence $(x_n)_{n\geq 1}$ to 0.
- Def. (mean-square convergence) When $p = 2$, the convergence is in the mean square (MS) sense.
  - The MS convergence is denoted by
  $$\underset{n\to\infty}{\text{l.i.m.}} \ X_n(s) = X(s)$$
  on $\mathcal{D}$, and reads, "the limit in the mean of $X_n(s)$ equals $X(s)$ on $\mathcal{D}$."
- $X_n(s) \to X(s)$ in the pth mean on $\mathcal{D}$ as $n \to \infty$
- $X_n(s) \to X(s)$ in the pth mean on $\mathcal{D}$.
- Lemma. X_n converges almost everywhere to X does not imply X_n converges in the pth mean to X. Conversely, X_n converges in the pth mean to X does not imply X_n converges almost everywhere to X.
- Lemma. For a sequence $(X_n(s))_{n\geq 1}$ of real functions, it converges in the MS

Ex. Later

when we talk about the convergence of a sequence of RVs.

sense iff the sequence is Cauchy, i.e.,

$$\forall \varepsilon > 0, \exists N(\varepsilon) > 0, \text{ s.t. } m,n \geq N(\varepsilon) \Longrightarrow \int_{s\in\mathcal{D}} |X_n(s) - X_m(s)|^2 ds \leq \varepsilon.$$

or simply

$$\int_{s\in\mathcal{D}} |X_n(s) - X_m(s)|^2 ds \to 0$$

as $n \to \infty$ and $m \to \infty$, or for sufficiently large m and n.

y_n

5. Def. (convergence in measure) A sequence $(X_n(s))_{n\geq 1}$ of real functions converges in measure to a real function $X(s)$ on $\mathcal{D}$ if

$$\forall \varepsilon > 0, \boxed{m(\{s \in \mathcal{D} : |X_n(s) - X(s)| > \varepsilon\})} \to 0$$

$\exists N(\varepsilon) \in \mathbb{N}$ s.t $m(\quad) \leq \varepsilon$ for $n \geq N(\varepsilon)$.

- Note that if we view $m(\{s \in \mathbb{R} : |X_n(s) - X(s)| > \varepsilon\})$ as $x_n(\varepsilon)$ then the convergence in measure is nothing but the convergence a sequence $(x_n(\varepsilon))_n$ to zero .
- When $m(\mathcal{D}) < \infty$, converges in measure iff $\forall \varepsilon > 0, m(\{s \in \mathcal{D} : |X_n(s) - X(s)| \leq \varepsilon\}) \to m(\mathcal{D})$.
- $X_n(s) \to X(s)$ in measure on $\mathcal{D}$ as $n \to \infty$

5. Def. (convergence in measure) A sequence $(X_n(s))_{n\geq 1}$ of real functions converges in measure to a real function $X(s)$ on $\mathcal{D}$ if

$$\forall \varepsilon > 0, \boxed{m(\{s \in \mathcal{D} : |X_n(s) - X(s)| > \varepsilon\})} \to 0$$

$\exists N(\varepsilon) \in \mathbb{N}$ s.t. $m(\quad) \leq \varepsilon$ for $n \geq N(\varepsilon)$.

- Note that if we view $m(\{s \in \mathbb{R} : |X_n(s) - X(s)| > \varepsilon\})$ as $x_n(\varepsilon)$ then the convergence in measure is nothing but the convergence a sequence $(x_n(\varepsilon))_n$ to zero .
- When $m(\mathcal{D}) < \infty$, converges in measure iff $\forall \varepsilon > 0, m(\{s \in \mathcal{D} : |X_n(s) - X(s)| \leq \varepsilon\}) \to m(\mathcal{D})$.
- $X_n(s) \to X(s)$ in measure on $\mathcal{D}$ as $n \to \infty$
- $X_n(s) \to X(s)$ in measure on $\mathcal{D}$
- Lemma. If X_n converges almost everywhere to X then X_n converges in measure to X.
- Lemma. If X_n converges in the pth mean to X then X_n converges in measure to X.