CM24-230518Thu

Announcements

1. Final Exam will be taken either
   a. half on 6/~~13~~ 6 and half on 6/~~15~~ 8 during the CM hours for 2x75min
   b. or on 6/~~14~~ 7 19:00-22:00.
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
   Passwords are ...
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
4. Topics of Voluntary Projects must be discussed with the Lecturer before you take the final exam.
5. HWs will be assigned. Check the PLMS.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

- Pa 9.1 Definitions
- Pe 6 Random Process--Temporal Characteristics
- Pa 7.4 Stochastic Convergence and Limit Theorems
- Pa 9.2 Systems with Stochastic Inputs
- Pa 9.3 The Power Spectrum
- Pe 7 Random Process--Spectral Characteristics
- Pe 8 Linear Systems with Random Inputs
- Pe 9 Optimal Linear Systems

Review

1. 4 modes of convergence
   a. almost sure
   b. in probability (measure)
   c. in the pth mean
   d. in distribution
      i. CDF
      ii. characteristic function
2. LLN
   a. strong : a.s.
   b. weak : in probability

Def. $X_n \to X$ in distribution if $F_{X_n}(x) \to F_X(x)$ on all points of continuity

Lemma. $X_n \to X$ in distribution iff $\Phi_{X_n}(\omega) \to \Phi_X(\omega)$ (for ~~all~~ $\omega \in \mathbb{R}$ ~~all~~ almost every)

$= E[e^{j\omega X_n}]$

$= \int \quad dP(s)$

Preview

1. CLT
2. Examples
   a. Monte-Carlo integration/simulation
   b. the Borel-Cantelli lemma to prove a convengence w.p.1
   c. Gaussian approximation to a binomial random variable
   d. Poisson approximation to a binomial random variable
   ← convergence in distribution
   e. BER of a synchronous DS/SSMA with random sequences
3. Summary
4. (Pa 9.5) Stochastic Calculus
5. (Pa 9.1) Systems with stochastic inputs

Central Limit Theorem (CLT)

- Simply put, a central limit theorem is about
  - the convergence of $\sqrt{n}$ times the arithmetic average of zero-mean random variables
  - to a Gaussian random variable
  - in distribution.
- Again, to define the arithmetic average, we first need a sequence of random variables $(X_i)_{i=1}^{\infty}$.
  - There are many different versions of CLTs.
  - Each CLT has a different set of restrictions on the random variables $X_1, X_2, \ldots$.
  - Thus, you must be very careful in invoking a CLT.
- Although
  - the exact CLT is to justify a Gaussian approximation to $\sqrt{n}$ times the arithmetic average of zero-mean random variables,
  - it is often used to justify a Gaussian approximation to the arithmetic average of nonzero-mean random variables.

$M_n \triangleq \frac{X_1 + \cdots + X_n}{n} \to \mu_X$ a.s. ← SLLN

$E[M_n] = \mu_X$. $Var(M_n) = E\left[\left\{\frac{(X_1-\mu_X)+\cdots+(X_n-\mu_X)}{n}\right\}^2\right] = \frac{\sigma^2}{n}$ in prob. ← WLLN

- Theorem. (a CLT) Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with mean μ_X and variance σ^2, and define

$$Z_n \triangleq \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu_X}{\sigma}.$$

$E\{Z_n\} = 0$

$Var(Z_n) = E\{Z_n^2\} = \frac{n \cdot 1}{n} = 1$

Then, the characteristic function $\Phi_{Z_n}(\omega)$ of Z_n converges to that of a Gaussian random variable with mean 0 and variance 1, i.e.,

$$\lim_{n \to \infty} \Phi_{Z_n}(\omega) = \exp\left(-\frac{\omega^2}{2}\right).$$

$Y \sim N(\mu, \sigma^2)$

$\Phi_Y(\omega) = e^{j\omega\mu - \frac{\sigma^2\omega^2}{2}}$

- Proof. Omitted.

- The above CLT says

$$\frac{X_1 + X_2 + \ldots X_n}{n} \approx Y \sim \mathcal{N}\left(\mu_X, \frac{\sigma^2}{n}\right).$$

" Z_n ~~is~~ can be well approximated by the standard Gaussian r.v. for s.l.n "

However, an LLN says

$$\frac{X_1 + X_2 + \ldots X_n}{n} \approx \mu_X.$$

Which approximation do you prefer?

$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \mu_X}{\sigma} = Z_n$ $N(0, 1)$

$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - \mu) \leftarrow N(0, \sigma^2)$

$\frac{1}{n} \sum_{i=1}^{n} \frac{X_i - \mu}{\sigma} = \frac{Z_n}{\sqrt{n}} \leftarrow N(0, \frac{1}{n})$

$\frac{1}{n} \sum_{i=1}^{n} \frac{X_i - \mu}{1} = \frac{\sigma Z_n}{\sqrt{n}} \leftarrow N(0, \frac{\sigma^2}{n})$

$M_n = \frac{X_1 + X_2 + \cdots + X_n}{n} = \frac{\sigma Z_n}{\sqrt{n}} + \mu \leftarrow N(\mu, \frac{\sigma^2}{n})$

- Theorem. (a CLT) If X_i's are i.i.d. random variables with $\Pr(X_i = -1) = \Pr(X_i = 1) = 1/2$, then

$$\frac{X_1 + X_2 + \ldots X_n}{\sqrt{n}} = Z_n$$

$E\{X_i\} = 0$, $Var\{X_i\} = 1$

$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \frac{X_i - \cancel{0}}{1} \to N(0, 1)$

converges in distribution to a Gaussian random variable with mean zero and variance 1.

- Proof. Since the characteristic function of $X_i/\sqrt{n}$ is

$$\Phi(\omega) = \cos\left(\frac{\omega}{\sqrt{n}}\right),$$

$\Phi_{X_i}(\omega) = E[e^{j\omega X_i}]$

$= e^{j\omega \cdot 1} \frac{1}{2} + e^{+j\omega(-1)} \cdot \frac{1}{2}$

$= \cos\omega$

the characteristic function of $(X_1 + X_2 + \ldots X_n)/\sqrt{n}$ is given by

$$\boxed{\Phi_n(\omega) = \left\{\cos\left(\frac{\omega}{\sqrt{n}}\right)\right\}^n.}$$

$\Phi_{Z_n}(\omega) = E[e^{j\omega \frac{1}{\sqrt{n}} \Sigma X_i}]$

$= E[e^{j\omega \frac{X_1}{\sqrt{n}}} e^{j\omega \frac{X_2}{\sqrt{n}}} \cdots e^{j\omega \frac{X_n}{\sqrt{n}}}]$

By using l'Hospital's rule, we can show that

$$\boxed{\ln \Phi_n(\omega) = \frac{\ln \cos\left(\frac{\omega}{\sqrt{n}}\right)}{\frac{1}{n}}}$$

converges to $-\omega^2/2$. Thus, by the continuity of the logarithmic function, we have

$$\ln \Phi_n(\omega) = \frac{\cdots}{\frac{1}{n}}$$

converges to $-\omega^2/2$. Thus, by the continuity of the logarithmic function, we have

$$\Phi_n(\omega) \longrightarrow \exp\left(-\frac{\omega^2}{2}\right).$$

Therefore, the conclusion follows.

$$= E\left[e^{j\omega\frac{X_1}{\sqrt{n}}} e^{j\omega\frac{X_2}{\sqrt{n}}} \cdots e^{j\omega\frac{X_n}{\sqrt{n}}}\right]$$

$$= \cdots = \underbrace{\Phi_{X_1}\left(\frac{\omega}{\sqrt{n}}\right) \times \cdots \Phi_{X_n}\left(\frac{\omega}{\sqrt{n}}\right)}$$

$$= \left\{\cos\left(\frac{\omega}{\sqrt{n}}\right)\right\}^n$$

$$\left[\begin{array}{l}\left(1+\frac{1}{x}\right)^x \to e \text{ as } x\to\infty \\ (1+x)^{\frac{1}{x}} \to e \text{ as } x \to 0\end{array}\right.$$

$$\lim_{n\to\infty} \left\{\cos\left(\frac{\omega}{\sqrt{n}}\right)\right\}^n$$

$\boxed{\lim_{n\to\infty} x_n \quad \text{if } f \text{ is continuous ft}}$

$$\lim_{n\to\infty} f(x_n) = f\left(\lim_{n\to\infty} x_n\right)$$

· Trick.

"$g(x)$ is continuous."

$$\equiv \boxed{\lim_{x\to x_0} g(x) = g\left(\lim_{x\to x_0} x\right) = g(x_0)}$$

So, instead of showing

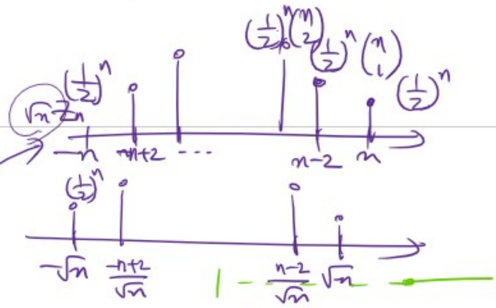
$$\left\{\cos\left(\frac{\omega}{\sqrt{n}}\right)\right\}^n \to \exp\left(-\frac{\omega^2}{2}\right) \qquad e^{\ln\{\cos(\frac{\omega}{\sqrt{n}})\}^n} \to e^{-\frac{\omega^2}{2}}$$

$$\ln\left\{\cos\left(\frac{\omega}{\sqrt{n}}\right)\right\}^n \to -\frac{\omega^2}{2} \qquad n \ln \cos\left(\frac{\omega}{\sqrt{n}}\right) = \frac{\ln\cos\left(\frac{\omega}{\sqrt{n}}\right)}{\frac{1}{n}}$$

l'Hospital's rule

$$= \frac{\frac{-\sin(\frac{\omega}{\sqrt{n}})}{\cos(\frac{\omega}{\sqrt{n}})}\left(-\frac{1}{2}\right) n^{-\frac{3}{2}} \omega}{-\frac{1}{n^2}} \qquad \frac{1}{\frac{n^{\frac{3}{2}}}{n^2}}$$

$$= \frac{\left(-\frac{1}{2}\right)}{\cos\left(\frac{\omega}{\sqrt{n}}\right)} \frac{\sin\left(\frac{\omega}{\sqrt{n}}\right)\omega}{\frac{1}{\sqrt{n}}} \to -\frac{\omega^2}{2}$$



☐ $F_{Z_n}(z)$

$$Z_n = \frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n}}$$

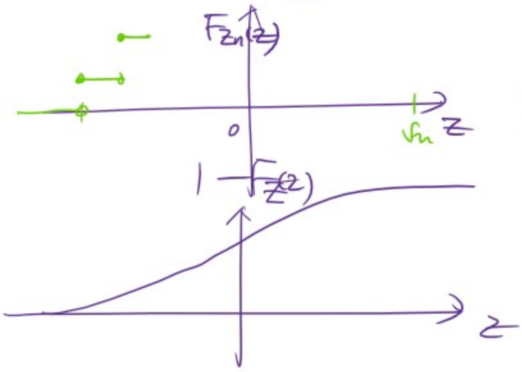
1 1

1 2 1

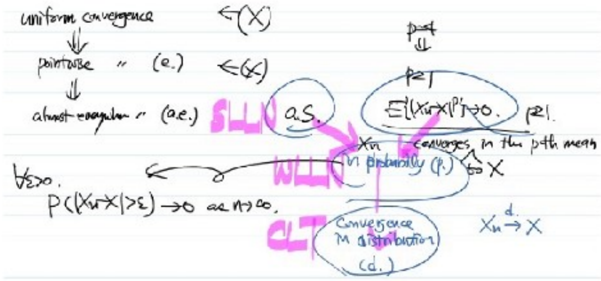
1 3 3 1

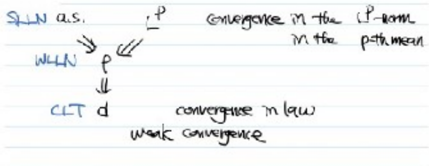
⋮

1 n n 1



$$F_Z(z) = P(Z \le z)$$
$$= 1 - Q(z)$$





Be careful. The arithmetic mean converges to 0 by an LLN. $\sqrt{n}$ times the arithmetic average converges to a Gaussian.

– This theorem is often used to justify the approximation of a binomial random variable $S_n \sim \mathcal{B}(n, 1/2)$ to the Gaussian random variable $N_n \sim \mathcal{N}(n/2, n/4)$.

  * Be careful. S_n does not converge because its mean diverges to infinity. N_n does not converge because its mean and variance diverge to infinity.
  * (Gaussian approximation to binomial) Similarly, we can invoke the approximation of a binomial random variable $S_n \sim \mathcal{B}(n, p)$ to the Gaussian random variable $N_n \sim \mathcal{N}(np, npq)$.

• (Poisson approximation to binomial) We've already seen a Poisson approximation to a binomial random variable, where the difference is on that np is fixed as λ when n tends to infinity.

$$\binom{n}{k} p^k (1-p)^{n-k} \approx \frac{(np)^k e^{-np}}{k!} \qquad n \text{ large and } p \text{ small}$$

~~$P(X_i=0)=P(X_i=1)=1/2$~~

$P(X_i=1)=p,\ P(X_i=0)=1-p$

X_i's i.i.d.

$S_n = X_1+\cdots+X_n \sim B(n,p)$

$E\{S_n\}=np$

$Var\{S_n\}=np(1-p)$

$\frac{S_n-np}{\sqrt{np(1-p)}} \to N(0,1)$

$S \sim N(np, np(1-p))$

$\Phi_{X_i}(\omega)=E[e^{j\omega X_i}]=e^{j\omega}p+e^{-j\omega}(1-p)=(e^{j\omega}-e^{-j\omega})p+e^{-j\omega}$

$\Phi_{S_n}(\omega)=(\quad)^n$

np fixed

take $n\to\infty$

Gaussian approximation to binomial

$Y_n \sim B(n, \frac{1}{2})$

$Y_n = X_1 + X_2 + \cdots + X_n = \sqrt{n} \cdot \frac{X_1+X_2+\cdots+X_n}{\sqrt{n}}$

X_i's are i.i.d, w/

$Pr(X_i=0) = Pr(X_i=1) = 1/2$

$Y_n - n/2 = \sqrt{n} \cdot \frac{(X_1-\frac{1}{2})+(X_2-\frac{1}{2})+\cdots+(X_n-\frac{1}{2})}{\sqrt{n}}$

$\sim N(0, \frac{n}{4})$

$Y_n \approx N(\frac{n}{2}, \frac{n}{4})$

- $B(n,p) \approx N(np, np(1-p))$

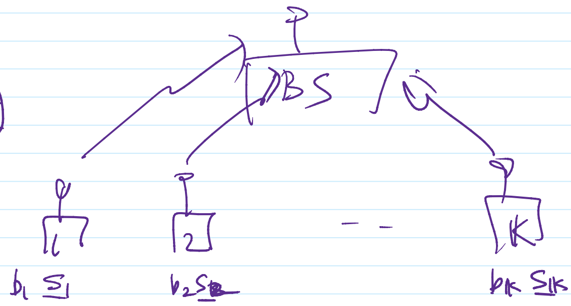
II Synchronous CDMA (DS/SSMA) with random sequences

$\underline{Y} = \sqrt{P}\sum_{k=1}^{K} b_k \underline{S}_k + \underline{N}$

$\{\underline{Y}=\underline{y}\}$, $\underline{y} \in \mathbb{R}^N$ $\underline{N} \sim N(\underline{0}, \sigma^2 I)$

$\{\underline{S}_k = \underline{s}_k\}$, $\underline{s}_k \in \{-1, 1\}^N$

$\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \\ \vdots \end{bmatrix}$



$E\{\ \} = 0$, $Var\{\ \} = N$.

A matched filter for the k'th user

$\underline{S}_{k'}^T \underline{Y} = \sqrt{P}\, b_{k'} N + \sqrt{P}\sum_{k \neq k'} b_k \underline{S}_{k'}^T \underline{S}_k + \underline{S}_{k'}^T \underline{N}$ $E[\underline{S}_{k'}^T \underline{N}] = 0$, $Var[\underline{S}_{k'}^T \underline{N}] = N\sigma^2$

$-N \quad 0 \quad N$

$Var\{\ \} = (K-1)NP$

w/o MAI

$P(b_{k'} \neq \hat{b}_{k'})$

$= P(b_{k'} \neq sign(\underline{S}_{k'}^T \underline{Y})) = P(\underline{S}_{k'}^T \underline{Y} < 0 \mid b_{k'} = 1)\, P(b_{k'}=1) + P(\underline{S}_{k'}^T \underline{Y} > 0 \mid b_{k'} = -1)\, P(b_{k'}=-1)$

$= Q\left(\frac{\sqrt{P} N \cdot 1}{\sqrt{N\sigma^2}}\right) = Q\left(\sqrt{\frac{PN}{\sigma^2}}\right)$

w/ MAI $E\{\ \} = 0$ $Var\{\ \} =$

BER

$\simeq Q\left(\frac{\sqrt{P} N}{\sqrt{N\sigma^2 + (K-1)NP}}\right) = Q\left(\sqrt{\frac{PN^{\cancel{2}}}{\cancel{N}(\sigma^2 + P(K-1))}}\right)$

$= Q\left(\sqrt{\frac{PN}{\sigma^2 + P(K-1)}}\right)$