2023년 5월 23일 화요일 12:38

CM25-230523Tue

Announcements

1. Final Exam will be taken either
   a. half on 6/6 and half on 6/8 during the CM hours for 2x75min
   b. or on 6/7 19:00-22:00. Wednesday
2. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
3. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
4. Topics of Voluntary Projects must be discussed with the Lecturer before you take the final exam.
5. HWs will be assigned. Check the PLMS.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

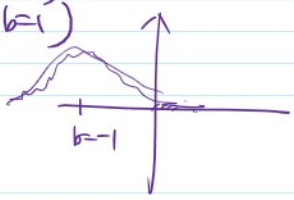
Pa 9.1 Definitions
Pe 6 Random Process--Temporal Characteristics
Pa 7.4 Stochastic Convergence and Limit Theorems
Pa 9.2 Systems with Stochastic Inputs
Pa 9.3 The Power Spectrum
Pe 7 Random Process--Spectral Characteristics
Pe 8 Linear Systems with Random Inputs
Pe 9 Optimal Linear Systems

a.s. (w.p.1) → in probability ← in the p-th mean, $p \geq 1$
in probability → in distribution

Review

1. CLT
2. Examples
   a. Gaussian approximation to a binomial random variable
   b. Poisson approximation to a binomial random variable
   c. BER of a synchronous ~~DS/SSMA~~ DS-CDMA with random sequences

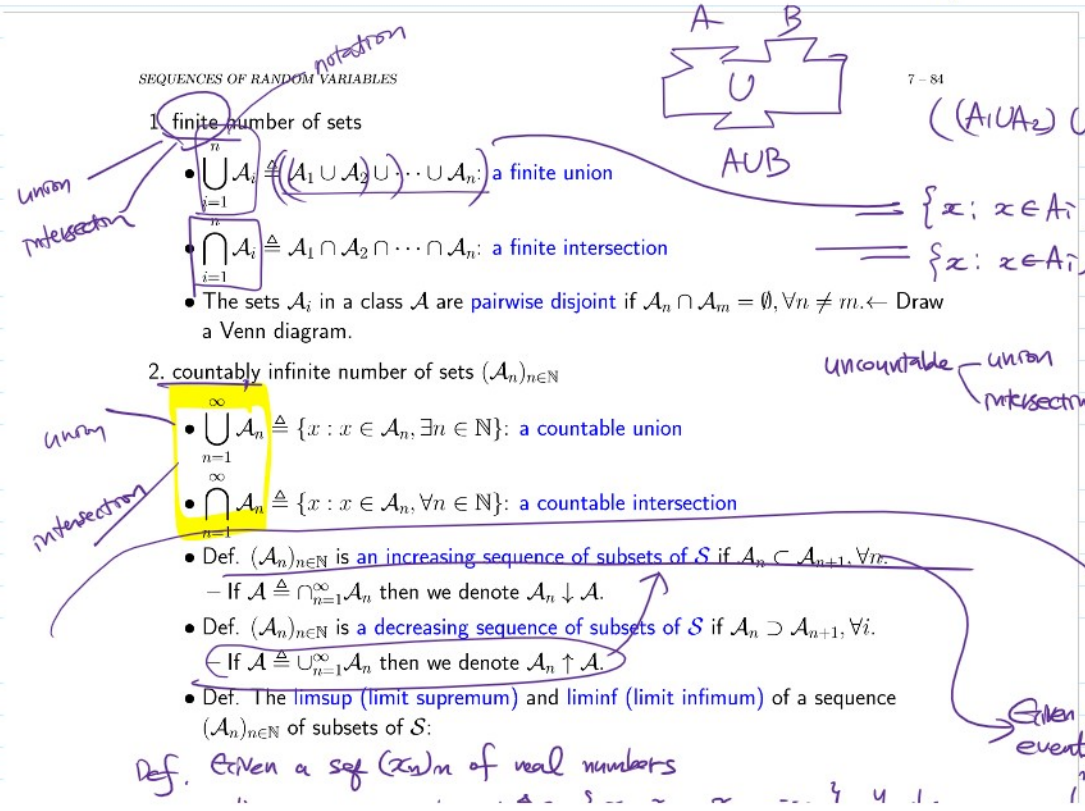
$$P_b = P(\hat{b} \neq b | b=-1)\,P(b=-1) + P(\hat{b} \neq b | b=1)\,P(b=1) = P(\hat{b} \neq b | b=-1) = P(y>0 | b=-1) \cong Q(\sqrt{\quad})$$



Preview

a. Monte-Carlo integration/simulation
b. the Borel-Cantelli lemma to prove a convengence w.p.1

1. Summary
2. (Pa 9.5) Stochastic Calculus
3. (Pa 9.1) Systems with stochastic inputs



$(\mathcal{A}_n)_{n\in\mathbb{N}}$ of subsets of $\mathcal{S}$:

- $\limsup_{n\to\infty} \mathcal{A}_n \triangleq \bigcap_{n=1}^{\infty}\bigcup_{m\geq n} \mathcal{A}_m = \{s \in \mathcal{S} : s \in \mathcal{A}_n \text{ for infinitely many } n\}$
- $\liminf_{n\to\infty} \mathcal{A}_n \triangleq \bigcup_{n=1}^{\infty}\bigcap_{m\geq n} \mathcal{A}_m = \{s \in \mathcal{S} : s \in \mathcal{A}_n \text{ for all but finitely many } n\}$
- $\cap_{n=1}^{\infty}\mathcal{A}_n \subset \liminf_{n\to\infty} \mathcal{A}_n \subset \limsup_{n\to\infty} \mathcal{A}_n \subset \cup_{n=1}^{\infty}\mathcal{A}_n$
- Def. $\lim_{n\to\infty} \mathcal{A}_n = \mathcal{A}$ if $\lim_{n\to\infty} 1_{\mathcal{A}_n}(s) = 1_{\mathcal{A}}(s), \forall s \in \mathcal{S}$.
  - $\lim_{n\to\infty} \mathcal{A}_n = \mathcal{A}$ iff $\limsup_{n\to\infty} \mathcal{A}_n = \liminf_{n\to\infty} \mathcal{A}_n = \mathcal{A}$

3. uncountably infinite number of sets $(\mathcal{A}_\alpha)_{\alpha\in\mathcal{I}}$, where $\mathcal{I}$ is an uncountable index set.
   - $\bigcup_{\alpha\in\mathcal{I}} \mathcal{A}_\alpha \triangleq \{x : x \in \mathcal{A}_\alpha, \exists\alpha \in \mathcal{I}\}$: an uncountable union
   - $\bigcap_{\alpha\in\mathcal{I}} \mathcal{A}_\alpha \triangleq \{x : x \in \mathcal{A}_\alpha, \forall\alpha \in \mathcal{I}\}$: an uncountable intersection
   - /ex/ When $\mathcal{A}_\alpha = [2, \alpha)$, find $\cap_{\alpha>3}\mathcal{A}_\alpha$
- Q. Given
$$A_n = \left\{\frac{0}{n}, \frac{1}{n}, \cdots, \frac{n^2}{n}\right\},$$
  - show that $\limsup_n A_n = \mathbb{Q}_+$ and
  - $\liminf_n A_n = \mathbb{Z}_+$

- Q. Given
$$B_n = \left[0, \frac{n}{n+1}\right],$$
  - show that $\limsup_n B_n = [0, 1)$, $\liminf_n B_n = [0, 1)$, and hence
  - $\lim_n B_n = [0, 1)$
  -

- Q. (5 points: 21-1 HW#2) When you repeatedly toss an unbiased coin independently, you stop at the first show of heads. Answer the following questions.
  (a) Classify the sample space.
  (b) Find the prob. that you stop at an even number of tosses.
  (c) Find the prob. that you stop at an odd number of tosses.
  (d) Find the prob. that you toss infinitely many times.
- Q. (17 points: 21-1 HW#2) We have a deck of cards, on the face of each card an integer is printed; thus the cards are $\{1, 2, 3...\}$. At the nth round of this game, the first n cards are taken, they are shuffled. You pick one of them. If your pick is 1, you win that round. Let $\mathcal{B}_n$ denote the event that you win the nth round. Each round is independent of all the other rounds. Let $\mathcal{B}_n = \{Y_n = 1\}$ and $\mathcal{B}_n^c = \{Y_n = 0\}$ and define $M = \sum_{n=1}^{\infty} Y_n$. Answer the following questions.
  (a) (2 points) Are $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n, ...$ independent events? Are $\mathcal{A}_1, \mathcal{A}_2, ..., \mathcal{A}_n, ...$ pairwise disjoint events?
  (b) (5 points) What is the meaning of $\cap_{n=1}^{\infty}\mathcal{A}_n$ and find $P(\cap_{n=1}^{\infty}\mathcal{A}_n)$.
  (c) (5 points) Find $P(M = 1)$.
  (d) (5 points) What is the meaning of $\cup_{n=1}^{\infty}\mathcal{A}_n$ and find $P(\cup_{n=1}^{\infty}\mathcal{A}_n)$.
  (e) (0 point) Show $P(M = m) = 0$ for all $m > 1$ and show $P(M = \infty) = 1$ if you can. (Hint: Borel-Cantelli lemma)

- (Wiki)
  – In probability theory, the Borel-Cantelli lemma is a theorem about sequences of events. In general, it is a result in measure theory.
  – It is named after Emile Borel and Francesco Paolo Cantelli, who gave statement to the lemma in the first decades of the 20th century. A related result, sometimes called the second Borel-Cantelli lemma, is a partial converse of the first Borel-Cantelli lemma.
  – The lemma states that, under certain conditions, an event will have probability of either zero or one. Accordingly, it is the best-known of a class of similar theorems, known as zero-one laws. Other examples include Kolmogorov's zero-one law and the Hewitt-Savage zero-one law.
  – Let $E_1, E_2, ...$ be a sequence of events in some probability space. The Borel-Cantelli lemma states:
    If the sum of the probabilities of the event E_n is finite

$$\sum_{n=1}^{\infty} \Pr(E_n) < \infty,$$

    then the probability that infinitely many of them occur is 0. That is,

$$\Pr\left(\limsup_{n\to\infty} E_n\right) = 0,$$

$= 0 \le P_n \le 1$

$\Pr(\{\limsup_{n\to\infty} E_n\}^c) = 1$ w.p.1

    or, equivalently, the probability that only finitely many of them occur is 1, that is,

$$\Pr\left(\liminf_{n\to\infty} E_n^c\right) = 1.$$

  – A related result, sometimes called the second Borel-Cantelli lemma, is a partial converse of the first Borel-Cantelli lemma. The lemma states:
    If the events E_n are independent and the sum of the probabilities of the E_n diverges to infinity, i.e.,

$$\sum^{\infty} \Pr(E_n) = \infty$$

or, equivalently, the probability that only finitely many of them occur is 1, that is,

$$\Pr\left(\liminf_{n\to\infty} E_n^c\right) = 1.$$

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then the probability that infinitely many of them occur is 1, i.e.,

$$\Pr(\limsup_{n\to\infty} E_n) = 1.$$

The assumption of independence can be weakened to pairwise independence, but in that case the proof is more difficult.

• Examples: We have a deck of cards, on the face of each card an integer is printed; thus the cards are $\{1, 2, 3...\}$.

Q1. At the nth round of this game, the first n^2 cards are taken, they are shuffled. You pick one of them.

∗ If your pick is 1, you win that round. Let $\mathcal{A}_n$ denote the event that you win the nth round.

∗ The complement $\mathcal{A}_n^c$ of $\mathcal{A}_n$ will represent that you lose the nth round.

∗ The event $\mathcal{A}(\triangleq \limsup_{n\to\infty} \mathcal{A}_n)$ represents those scenarios in which you win infinitely many rounds.

∗ The complement of this event is $\mathcal{A}^c(\triangleq \liminf_{n\to\infty} \mathcal{A}_n^c)$, and this represents those scenarios in which you ultimately lose all of the rounds, i.e., you win only finitely many rounds.

∗ Find $P(\mathcal{A}) = 1 - P(\mathcal{A}^c)$.

– A2. $P(\limsup_{n\to\infty} \mathcal{A}_n) = 0$ and $P(\liminf_{n\to\infty} \mathcal{A}_n^c) = 1$. Thus, a player of this game will deterministically experience that there comes a time, after which he never wins.

Q2. At the nth round of this game, the first n cards are taken, they are shuffled. You pick one of them.

* If your pick is 1, you win that round. Let $\mathcal{B}_n$ denote the event that you win the nth round.
* The complement $\mathcal{B}_n^c$ of $\mathcal{B}_n$ will represent that you lose the nth round.
* The event $\mathcal{B}(\triangleq \limsup_{n\to\infty} \mathcal{B}_n)$ represents those scenarios in which you win infinitely many rounds.
* The complement of this event is $\mathcal{B}^c(\triangleq \liminf_{n\to\infty} \mathcal{B}_n^c)$, and this represents those scenarios in which you ultimately lose all of the rounds, i.e., you win only finitely many rounds.
* Find $P(\mathcal{B}) = 1 - P(\mathcal{B}^c)$.

– A2. $P(\limsup_{n\to\infty} \mathcal{B}_n) = 0$ and $P(\liminf_{n\to\infty} \mathcal{B}_n^c) = 1$. Thus, a player of this game will always win infinitely many times.

***Unequal Distributions**

- A sufficient condition for the theorem's validity

$$\sigma_{X_i}^2 > B_1 > 0 \qquad i = 1, 2, ..., n \tag{7.4-1a}$$
$$\mathbf{E}[|X_i - \mu_i|^3] < B_2 \qquad i = 1, 2, ..., n \tag{7.4-1b}$$

- No one random variable in the sum dominates.
- The CLT guarantees only that the *distribution* of the sum becomes gaussian.
- not necessarily the pdf
- quite accurate for relatively small values of n in the central region of the gaussian curve near the mean

***Equal Distributions**

- independent and identically distributed (i.i.d.) random variables
- continuous
- We shall work with the zero-mean, unit-variance random variable

$$W_n = (Y_n - \bar{Y_n})/\sigma_{Y_n} = \sum_{i=1}^{n}(X_i - \mu_i) \Big/ \left[\sum_{i=1}^{n}\sigma_{X_i}^2\right]^{1/2}$$
$$= \frac{1}{\sqrt{n}\sigma_X}\sum_{i=1}^{n}(X_i - \bar{X}) \tag{7.4-2}$$

$$\mu_i = \bar{X} \qquad \text{all } i \tag{7.4-3}$$
$$\sigma_{X_i}^2 = \sigma_X^2 \qquad \text{all } i \tag{7.4-4}$$

- We show the characteristic function of W_n converges to that of a zero-mean, unit-variance gaussian random variable, i.e.,

$$\lim_{n\to\infty}\Phi_{W_n}(\omega) = \exp(-\omega^2/2). \tag{7.4-5}$$

- the uniqueness of Fourier transforms

$$\Phi_{W_n}(\omega) = \mathbf{E}[e^{j\omega W_n}] = E\left[\exp\left\{\frac{j\omega}{\sqrt{n}\sigma_X}\sum_{i=1}^{n}(X_i - \bar{X})\right\}\right]$$
$$= \left\langle E\left\{\exp\left[\frac{j\omega}{\sqrt{n}\sigma_X}(X_i - \bar{X})\right]\right\}\right\rangle^n \tag{7.4-6}$$

$$E\left\{\exp\left[\frac{j\omega}{\sqrt{n}\sigma_X}(X_i - \bar{X})\right]\right\}$$
$$= E\left\{1 + \left(\frac{j\omega}{\sqrt{n}\sigma_X}\right)(X_i - \bar{X}) + \left(\frac{j\omega}{\sqrt{n}\sigma_X}\right)^2\frac{(X_i - \bar{X})^2}{2} + \frac{R_n}{n}\right\}$$
$$= 1 - (\omega^2/2n) + \mathbf{E}[R_n]/n \tag{7.4-7}$$

- $\mathbf{E}[R_n]$ approaches zero as $n \to \infty$.

$$\ln[\Phi_{W_n}(\omega)] = n\ln\{1 - (\omega^2/2n) + \mathbf{E}[R_n]/n\} \tag{7.4-8}$$

- Since

$$\ln(1 - z) = -\left[z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right] \qquad |z| < 1 \tag{7.4-9}$$

$$\ln[\Phi_{W_n}(\omega)] = -(\omega^2/2) + \mathbf{E}[R_n] - \frac{n}{2}\left[\frac{\omega^2}{2n} - \frac{\mathbf{E}[R_n]}{n}\right]^2 + \cdots \tag{7.4-10}$$

$$\lim_{n\to\infty}\{\ln[\Phi_{W_n}(\omega)]\} = \ln\left\{\lim_{n\to\infty}\Phi_{W_n}(\omega)\right\} = -\omega^2/2 \tag{7.4-11}$$

$$\lim_{n\to\infty}\Phi_{W_n}(\omega) = e^{-\omega^2/2} \tag{7.4-12}$$

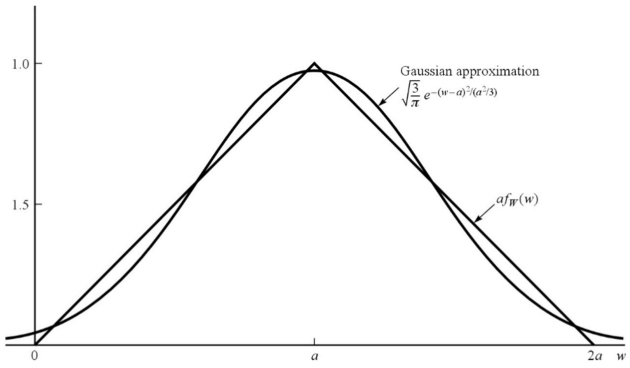


Figure 7.4-1

7.5 Random Numbers: Meaning and Generation

- (Monte Carlo integration) Suppose we want to compute

$$I = \int_0^1 g(x)dx,$$

where

  - it is not difficult to compute $g(x_0)$ for $0 \le x_0 \le 1$
  - but it is hard to directly compute the integration.
  - The trick is to view I as

$$I = \mathbf{E}\{g(X)\},$$

where X is a uniform random variable.

  - Now, we invoke an SLLN, which says

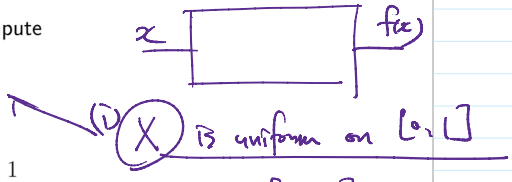
$$\frac{\sum_{i=1}^n g(X_i)}{n} \longrightarrow \mathbf{E}\{g(X)\}.$$

  - Thus, we can claim that

$$I \approx \frac{\sum_{i=1}^n g(X_i)}{n}$$

for sufficiently large n.

  - Therefore, instead of direct integration, we can perform a repeated trial with i.i.d. uniform random variables and obtain a good estimate of I.



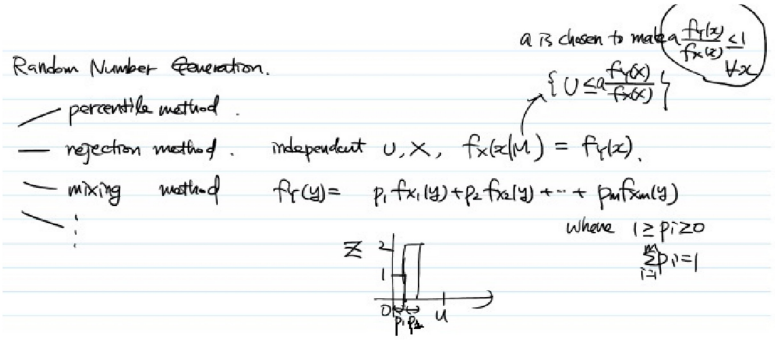
- Such a technique to replace an integration is called **a Monte Carlo integration**.

- (BER of BPSK in a Rayleigh fading channel) Q. Find a good estimate of

$$\mathbf{E}\left[Q\left(\sqrt{\frac{2E_b}{N_0}}X\right)\right],$$

  where X is a Rayleigh random variable with unit mean.
  - Actually, a closed-form expression can be obtained by direct integration. (Try it!)
  - However, a Monte Carlo integration can also be used to obtain an estimate. (Try it!)
- The trouble in using a Monte Carlo integration is on that there is few random number generator.
  - ERNIE (Electronic Random Number Indicating Equipment) 1 was a hardware random number generator based on quantum physics. It weas developled in 1957.
  - Mostly, people use a soft-ware pseudo-random number generator.
- (p. 287) The most general algorithm for generating a random number sequence is to use a nonlinear recursion.
  - MATLAB also uses pseudo-random number generators.
- methods to obtain arbitrary distribution
  - percentile transformation method

– rejection method
– mixing method



• Markov Chain Monte Carlo (MCMC)

5.7 SAMPLING AND SOME LIMIT THEOREMS

• sampling
  – continuous-time to discrete-time conversion = analog to digital conversion
  – continuous-time to vector conversion
  – discrete-time to discrete-time conversion
  – discrete-time to vector conversion
• Here, sampling has a different meaning.

Sampling and Estimation

• measuring some quantity
• We sample to get our measurement.
• an observation
• an unknown parameter
• an estimate
• How to optimally estimate the parameter using the observation?
  → statistical estimation theory ⊂ statistical signal processing

- Modes of convergence of a sequence of random variables

  (Wiki) Convergence of random variables

  – the probability space $(\mathcal{S}, \mathcal{A}, P)$

  – a sequence of random variables $X_1(s), X_2(s), ...$

  – the limit random variable $X(s)$

  – Def. $X_n \to X$ almost surely (a.s.), if

  $$P(\{s : \lim_{n\to\infty} X_n(s) = X(s)\}) = 1$$

  – Def. $X_n \to X$ in the pth mean ($p \geq 1$), if

  $$\lim_{n\to\infty} \mathbf{E}[|X_n - X|^p] = 0$$

  – Def. $X_n \to X$ in probability, if $\forall \epsilon > 0$,

  $$\lim_{n\to\infty} P(|X_n - X| \geq \epsilon) = 0$$

  – Def. $X_n \to X$ in distribution, if

  $$\lim_{n\to\infty} \Phi_{X_n}(\omega) = \Phi_X(\omega)$$

  Remarks:

  a.s. $\Rightarrow$ in probability

  L^p $\Rightarrow$ in probability

  in probability $\Rightarrow$ in distribution