2023년 5월 23일 화요일 12:44

CM26-230525Thu

Announcements

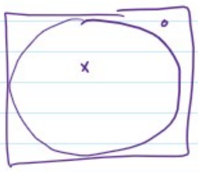
1. Final Exam will be taken on Wednesday 6/7 19:00-22:00 at LG 104.
2. Topics of Voluntary Projects must be discussed with the Lecturer before you take the final exam.
3. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm . Passwords are ...
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.
5. HWs will be assigned. Check the PLMS.

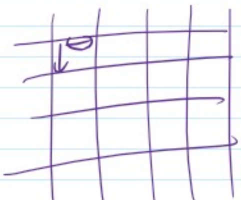
Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

$\cancel{\limsup_{n\to\infty} x_n}$ limsup $\boxed{\limsup_{n\to\infty}} x_n$ $\boxed{\liminf_{n\to\infty}} x_n$

Overview w/ Chapter numbers from the main textbook by Papoulis, (2)-(7) from Peebles

- Pa 9.1 Definitions
- Pe 6 Random Process--Temporal Characteristics
- Pa 7.4 Stochastic Convergence and Limit Theorems
- Pa 9.2 Systems with Stochastic Inputs
- Pa 9.3 The Power Spectrum
- Pe 7 Random Process--Spectral Characteristics
- Pe 8 Linear Systems with Random Inputs
- Pe 9 Optimal Linear Systems





0.1

$\frac{\# \text{ of}}{\# \text{ of throws}}$

$P\left(\liminf_{n\to\infty} E_n^c\right) = 1$

$P\left(\limsup_{n\to\infty} E_n\right) = 0$

a.s. convergence

Review

1. a limit point of a sequence of real numbers (x_n)
2. the limsup and the liminf of a sequence of real numbers
3. the limsup and the liminf of a sequence of events
4. the Borel-Cantelli lemma to prove a convengence w.p.1
5. Monte-Carlo integration/simulation
6. Summary of 4 modes of the convergence of a sequence of random variables

$x \in [0,1]$

$$\int_0^1 g(x)dx = \int_0^1 g(x) \underbrace{f_X(x)}_{=1 \text{ on}} dx = E\{g(X)\} \cong \frac{g(X_1)+\cdots+g(X_n)}{n}$$

Buffon's needle

Preview

1. (Pa 9.5) Stochastic Calculus
2. (Pa 9.1) Systems with stochastic inputs

Chapter 9
GENERAL CONCEPTS

Handout by Joon Ho Cho
based on A. Papoulis & S. U. Pillai, *Probability, Random Variables and Stochastic Processes, 4th ed.*

EECE 574, Spring 2023

/ex/

/ex/ $x(t) = \frac{t^2-1}{t-1}$

$\lim_{t\to 1} x(t) = 2$.

Def. $x: \mathbb{R}\to\mathbb{R}$ is continuous at t_0

if $\boxed{\lim_{t\to t_0} x(t) = x\left(\lim_{t\to t_0} t\right)} = x(t_0)$

Def. $x: \mathbb{R}\to\mathbb{R}$ is continuous

if x is " for all $t\in\mathbb{R}$

Def. Given $x: \mathbb{R}\to\mathbb{R}$, we say $\boxed{\lim_{t\to t_0} x(t) = c}$

if $\forall \varepsilon > 0$, $\exists\, \delta(\varepsilon) > 0$ s.t.

$|t - t_0| < \delta(\varepsilon) \Rightarrow |x(t) - c| < \varepsilon$.

$x(t)$ is well approximated by c for sufficiently close t to t_0.

/ex/ $x(t) = \frac{1}{t}$ $\lim_{t\to 0} x(t)$ does Not exist.

9.5 Continuity, Differentiation, Integration

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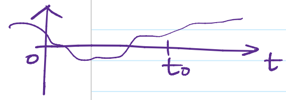
• ~~Q1 What is $X(t_0)$ given a RP $X(t)$? A1. a RV.~~



• Q2 What is

$$\lim_{t\to t_0} X(t)$$

?

$\cong$ "what is $X(t)$ for sufficiently close t to t_0?"

– Since a RP is a collection of RVs parameterized by t, it means $\boxed{\lim_{t\to t_0} \boxed{X(t,s)}}$ $\boxed{X_t(s)}$

– If the limit exists in some sense, then it must be a RV.

– We need to use the notion of the convergence of a sequence of RVs to a RV.

$\boxed{(S,\mathcal{F},P)} \xrightarrow{s} \boxed{X_t(s)} \rightarrow X_t$

• What do we mean by the continuity

$$\boxed{\lim_{t\to t_0} X(t)} = X(t_0)$$

?

$\lim_{t\to t_0} X_t(s) = X_{t_0}(s)$ 4 modes of the Convergence of a sequence of RVs

– a.s.

– in the pth mean

– in probability

– in distribution

$P(X(t_0) < x)$

$\simeq P(\lim_{t\to t_0} X(t) < x) \simeq \lim_{t\to t_0} P(X(t) < x)$

$P(X(t+\varepsilon) < x)$

for s. small ε . ? –

• We mainly consider the notion of the limit in the mean-square sense, i.e., we are interested in the case with

$$\text{l.i.m.}_{t\to t_0} X(t) = X(t_0).$$

$$\boxed{\boxed{\text{l.i.m.}_{t\to t_0} X(t)} = X(t_0)}$$

→ Def. $E[|X(t_0+\varepsilon) - X(t_0)|^2] \to 0$ as $\varepsilon \to 0$

Mean-Square Calculus

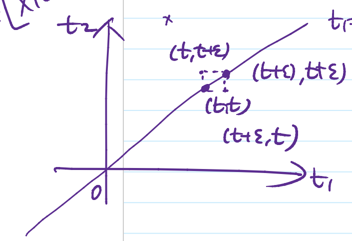
• From Chapter 9 of Stark and Woods, *Probability, Random Processes, and Estimation Theory for Engineers*, 2nd ed.

• Def. A random process $X(t)$ is continuous in the mean-square sense at the point t if

→ Def.) $E[|X(t+\epsilon) - X(t)|^2] \to 0$, as $\epsilon \to 0$. $\forall t$

• Def. A random process $X(t)$ is mean-square (m.-s.) continuous if $X(t)$ is continuous in the mean-square sense for all t.

$\triangleq E[X(t_1)X(t_2)]$



– $\boxed{\text{Theorem.}}$ The random process $X(t)$ is m.-s. continuous at t iff $R_{XX}(t_1,t_2)$ is continuous at the point $t_1 = t_2 = t$.

proof.

$R_{XX}(0) - R_{XX}(-\varepsilon) - R_{XX}(\varepsilon) + R_{XX}(0)$

$$E[|X(t+\epsilon)-X(t)|^2] = R_{XX}(t+\epsilon,t+\epsilon) - R_{XX}(t,t+\epsilon) - R_{XX}(t+\epsilon,t) + R_{XX}(t,t)$$

Clearly the right-hand side goes to zero as $\epsilon \to 0$ if the deterministic function R_{XX} is continuous at $t_1 = t_2 = t$. The only if part is somewhat complicated and will be omitted.

– $\boxed{\text{Corollary.}}$ A WSS random process is m.-s. continuous for all t iff $R_{XX}(\tau)$ is continuous at $t = 0$.

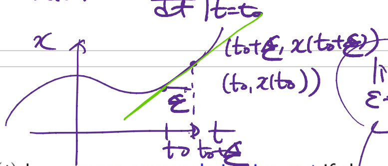
((S)) • Given a CT signal $x(t)$, what do we mean by $\dot{x}(t)$ $\frac{d}{dt}x(t)$?

$\lim_{n\to\infty}$

$\lim_{n\to\infty} \frac{m}{m+n}$

$\dot{x}(t_0) \leftarrow$ $\dot{x}(t_0)$ $\left.\frac{dx}{dt}\right|_{t=t_0}$

$\dot{x}(t) \leftarrow$



$\lim_{\varepsilon\to 0} \frac{x(t_0+\varepsilon) - x(t_0)}{(t_0+\varepsilon) - (t_0)} \triangleq \dot{x}(t_0)$

$y(\varepsilon)$ if exists

$\neq \lim_{n\to\infty} \lim_{m\to\infty} \frac{m}{m+n}$

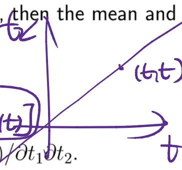
• Def. The random process $X(t)$ has a mean-square derivative at t if the mean-square limit of $[X(t+\epsilon) - X(t)]/\epsilon$ exists as $\epsilon \to 0$.

Def. $\dot{x}(t)$ is the derivative of $x(t)$

– Theorem. A random process $X(t)$ with autocorrelation function $R_{XX}(t_1,t_2)$ has a m.-s. derivative at time t iff $\partial^2 R_{XX}(t_1,t_2)/\partial t_1 \partial t_2$ exists at $t_1 = t_2 = t$.

– Theorem. If a random process $X(t)$ with mean function $\mu_X(t)$ and correlation function $R_{XX}(t_1,t_2)$ has an m.-s. derivative $X'(t)$, then the mean and correlation function of $X'(t)$ are given by

$E\left[\left|\frac{X(t+\varepsilon)-X(t)}{\varepsilon} - X'(t)\right|^2\right] \to 0$ as $\varepsilon \to 0$. $\forall t$



$$\boxed{E[X'(t)] = \mu_{X'}(t)} = d\mu_X(t)/dt \quad \boxed{\frac{d}{dt}E[X(t)]}$$

Def. $\dot{X}(t_0) = \lim_{\varepsilon\to 0} \frac{X(t_0+\varepsilon) - X(t_0)}{\varepsilon}$?

and

$$\boxed{R_{X'X'}(t_1,t_2)} = \partial^2 R_{XX}(t_1,t_2)/\partial t_1 \partial t_2.$$

– Theorem. The m.-s. derivative of a WSS random process $X(t)$ exists at t if the

$X'(t_0) = \dot{X}(t_0) = \text{l.i.m.}\ \frac{X(t_0+\varepsilon) - X(t_0)}{\varepsilon}$

and

$$\boxed{R_{X'X'}(t_1,t_2)} = \partial^2 R_{XX}(t_1,t_2)/\partial t_1 \partial t_2.$$

$E[\frac{d}{dt}X(t)]$ $\to$ t_1 ?

– Theorem. The m.-s. derivative of a WSS random process $X(t)$ exists at t if the autocorrelation function $R_{XX}(\tau)$ has derivatives up to order two at $\tau = 0$.

$$X'(t_0) = \dot{X}(t_0) = \underset{\varepsilon \to 0}{\mathrm{l.i.m.}} \frac{X(t_0+\varepsilon) - X(t_0)}{\varepsilon}.$$

/ex/

$$x(t) = \begin{cases} 1, & t \in \mathbb{Q}^c \cap [0,1] \\ 0, & t \in \mathbb{Q} \cap [0,1] \end{cases}$$

Q. $\int_0^1 x(t)\,dt$ $\quad 0 = t_0 < t_1^{(n)} < t_2^{(n)} \cdots < t_n^{(n)} = 1$

$$\underline{\int_0^1} x(t)\,dt \overset{?}{=} \overline{\int_0^1} x(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i^{(n)} \max_{t \in I_i^{(n)}} \{x(t)\}$$

((S)) $\int_{T_1}^{T_2} x(t)\,dt$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \Delta_i^{(n)} \min_{t \in I_i^{(n)}} \{x(t)\}$$

the Lebesgue integral.