2023년 5월 23일 화요일 12:44

CM28-230601Thu
Announcements
1. Final Exam will be taken on Wednesday 6/7 19:00-22:00 at LG 104.
2. Topics of Voluntary Projects must be discussed with the Lecturer before you take the final exam.
3. Handouts, annotated handouts are available at http://cisl.postech.ac.kr/class/eece574/schedule.htm .
   Passwords are ...
4. Lecture videos are available at cisl.postech YouTube channel. Clickable time stamps, please.

Corrections; Clarifications; Common mistakes,misunderstandings, misconceptions, myth; Frequently Asked Questions; Conventional wisdom

Overview
- Pa 9.1 Definitions
- Pe 6 Random Process--Temporal Characteristics
- Pa 7.4 Stochastic Convergence and Limit Theorems
- Pa 9.2 Systems with Stochastic Inputs
- Pa 9.3 The Power Spectrum
- Pe 7 Random Process--Spectral Characteristics
- Pe 8 Linear Systems with Random Inputs
- Pe 9 Optimal Linear Systems

Review
1. Stochastic Calculus
   a. MS Calculus
      i. continuity
      ii. differentiation
      iii. integration
   b. LCCDE

Preview

디지털 통신 시스템: 변복조와 전력 스펙트럼, K-MOOC, 2017.

http://cisl.postech.ac.kr/class/eece575/schedule.htm

← → C ▲ 주의 요함 | cisl.postech.ac.kr/class/eece575/schedule.htm

2021 2022 2023 youtube

Announcements Syllabus **Schedule/Downloads**

The schedule is tentative and subject to change. (last update on 2022-06-20)

Chapter		Sections	22-2				
			Handout	Week	Date and Lecture YouTube Link	Annotated PDF of Handout	
0	Syllabus						
1	Introduction						
2	Deterministic and Random Signal Analysis						
3	Digital Modulation Schemes						
4	Optimum Receivers for AWGN Channels						
5	Carrier and Symbol Synchronization						
6	An Introduction to Information Theory						
7	Linear Block Codes						
8	Trellis and Graph Based Codes						
9	Digital Communications Through Band-Limited Channels						
10	Adaptive Equalization						
11	Multichannel and Multicarrier Systems						
12	Spread Spectrum Signals for Digital Communications						
13	Fading Channels I: Characterization and Signaling						
14	Fading Channels II: Capacity and Coding						
15	Multiple Antenna Systems						
16	Multiuser Communications						

2018-2			KMOOC Week	KMOOC	YouTube Link	Topics and pdf Links
Week	Date	Meeting Number		Meeting #		
	Sept. 4	1		1	01	Analog vs. Digital Communications

• LCCDE

§ 9-2-3 Differential Equations

$x(t) \rightarrow$ LCCDE $\rightarrow y(t)$

$X(t) \rightarrow \quad \rightarrow Y(t)$

$$\boxed{\sum_{k=0}^{M} a_k y^{(k)}(t) = \sum_{\ell=0}^{N} b_\ell x^{(\ell)}(t)}$$

p q

Q1. Is this system LTI?

A1. No. in general.

Yes if initial rest (zero input until zero output until) (no zero-input response)

Q2. Any h(t) can be synthesized by an LCCDE?

A2. No, in general. For s.l. M & N, approximately ≃.

ARMA (p,q)

A special case of AR(p)

$$X(t_1) \quad \sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_1) = X(t_2) \quad \cdots (*)$$

① $\mu_Y(t) =$

$$E\left[\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t)\right] = E[X(t)]$$

$$E\left[\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{dt^\ell} Y(t)\right] = \boxed{\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{dt^\ell} \mu_Y(t) = \mu_X(t)}$$

given

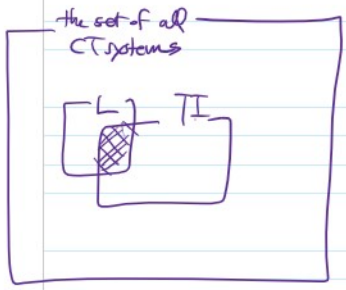
(ii) $R_{YY}(t_1, t_2) = ?$

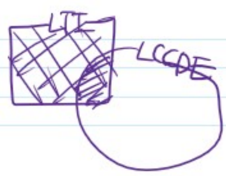
R_{XY} or R_{YX}

R_{YY}

$$E\left\{\left(\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_1) - X(t_1)\right) Y(t_2)\right\}$$

$Y(t_2)$





$E\{(\sum_{\ell=0} a_\ell Y^{(\ell)}(t) = X(t)) \cdots\}$

$Y(t)$ $\qquad \delta^{(0)}(t) = \delta(t)$

□ $X(t_1) \left(\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_2) = X(t_2) \right) \cdots (*)$

$x(t) \rightarrow \boxed{\frac{d}{dt}} \rightarrow \frac{dx(t)}{dt} \equiv x(t) \rightarrow \boxed{\delta^{(1)}(t)} \rightarrow \dot{x}(t)$

(ii) $R_{YY}(t_1,t_2) =$

General $X(t)$	WSS $X(t)$
$X(t_1) \sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_2) = X(t_1)X(t_2)$	$a_0 \mu_Y = \mu_X$
$\boxed{\sum_{\ell=0}^{p} a_\ell R_{XY^{(\ell)}}(t_1,t_2) = R_{XX}(t_1,t_2)}$	
$Y(t) \rightarrow \boxed{\delta^{(\ell)}(t)} \rightarrow Y^{(\ell)}(t)$ given	$\tau = t_1 - t_2$
$R_{XY}(t_1,t_2)' \Rightarrow \boxed{\sum_{\ell=0}^{p} a_\ell \frac{\partial^\ell}{\partial t_2^\ell} R_{XY}(t_1,t_2) = R_{XX}(t_1,t_2)}$ solve R_{XY}	$-\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{d\tau^\ell} R_{XY}(\tau) = R_{XX}(\tau)$
initial ______ Then,	
$\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_1) Y(t_2) = X(t_1) Y(t_2)$	
$\sum_{\ell=0}^{p} a_\ell \frac{\partial^\ell}{\partial t_1^\ell} R_{YY}(t_1,t_2) = R_{XY}(t_1,t_2)$	$\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{d\tau^\ell} R_{YY}(\tau) = R_{XY}(\tau)$
initial ______ Solve for R_{YY}	

Vector Processes and Multiterminal Systems

§9-2-4 Vector-valued r.p & ~~Multiterminal~~ multi-channel Systems. Multiple-Input Multiple-Output (MIMO)

$X(t) \rightarrow \boxed{\text{MIMO system}} \rightarrow Y(t)$

$X(t) = \begin{bmatrix} X_1(t) \\ \vdots \\ X_M(t) \end{bmatrix}$ $\qquad (\Omega, \mathcal{F}, P) \xrightarrow{\omega} \boxed{X_1(t,\omega)},\ \boxed{X_2(t,\omega)},\ \dots,\ \boxed{X_M(t,\omega)}$

□ MIMO LTI System

$$\underset{N\times 1}{y(t)} = \underset{N\times M}{H(t)} * \underset{M\times 1}{x(t)} = \int_{-\infty}^{\infty} H(t-\tau)\, x(\tau)\, d\tau = \int_{-\infty}^{\infty} H(\tau)\, x(t-\tau)\, d\tau$$

□ $X(t)$ is WSS

(i) $\mu_X(t) \triangleq E\{X(t)\} = \mu_X, \forall t$

(ii) $R_{XX}(t_1,t_2) \triangleq E\{X(t_1) X^H(t_2)\} = R_{XX}(\tau), \forall \tau = t_1 - t_2$

Linear: $Y(t) = \int_{-\infty}^{\infty} H(t,\tau)\, x(\tau)\, d\tau$

LTI: $Y(t) = \int_{-\infty}^{\infty} H(t-\tau)\, x(\tau)\, d\tau = H(t) * x(t)$

Alternative. M Gumm.

$r(t) = \int_{-\infty}^{\infty} C(t,\tau)\, s(t-\tau)\, d\tau$

$X_i(t)$ & $X_j(t)$ are jointly WSS!

□ If $X(t)$ is WSS, then $Y(t) = H(t) * X(t)$ is WSS.

$\mu_Y(t) = \cdots = \mu_Y, \forall t$

$R_{YY}(t_1,t_2) = \cdots = R_{YY}(t_1-t_2), \forall t_1,t_2$

/proof/ $E[Y(t)] = \begin{bmatrix} E\{Y_1(t)\} \\ \vdots \\ E\{Y_N(t)\} \end{bmatrix} = \begin{bmatrix} E\{\sum_{j=1}^{M} [H(t)]_{1,j} * [X(t)]_j\} \\ \vdots \\ E\{\sum_{j=1}^{M} [H(t)]_{N,j} * [X(t)]_j\} \end{bmatrix}$

$\mu_X \rightarrow \boxed{h(t)} \rightarrow$

$\int_{-\infty}^{\infty} h(\tau) \mu_X(t-\tau) d\tau$ $\qquad = H(t) * \mu_X(t)$

$\mu_X \int_{-\infty}^{\infty} h(\tau) d\tau \Rightarrow \mu_Y = \left(\int_{-\infty}^{\infty} H(t)\, dt\right) \mu_X$

$[E\{Y(t_1) Y^H(t_2)\}]_{i,j} = E\{\sum_{k=1}^{M} ([H(t_1)]_{i,k} * [X(t_1)]_k) (\sum_{k'=1}^{M} [H(t_2)]_{j,k'} * [X(t_2)]_{k'})^*\}$

$R_{YY}(t_1,t_2) = H(t_1) * R_{XX}(t_1,t_2) * H^H(t_2)$. $\qquad t_1 - t_2 = \tau$

$R_{YY}(t_1,t_2) = H(\tau) * R_{XX}(\tau) * H^H(-\tau)$.

$[E\{Y(t_1)Y(t_2)\}]_{i,j} = [\sum_{k=1} ([H(t)]_{i,k} * [X(t_1)]_k) \sum_{l=1} ([H(t)]_{j,l} * [X(t_2)]_l)]$

$R_{YY}(t_1,t_2) = H(t_1) * R_{XX}(t_1,t_2) * H^H(t_2)$. $\quad t_1 - t_2 = \tau$

$R_{YY}(t_1,t_2) = H(\tau) * R_{XX}(\tau) * H^H(-\tau)$.

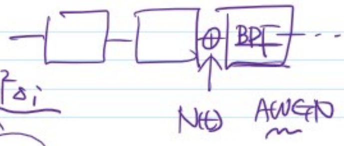
$R_{XY}(t_1,t_2) = R_{XX}(t_1,t_2) * H^H(t_2)$

9.3 The Power Spectrum

See [디지털 통신 시스템: 변복조와 전력 스펙트럼](http://cisl.postech.ac.kr/class/eece575/schedule.htm), K-MOOC, 2017.

http://cisl.postech.ac.kr/class/eece575/schedule.htm

- Preview.
  - Energy spectral density (ESD) and power spectral density (PSD) of deterministic CT signals
  - PSD of a CT random process
  - cross PSD of two CT random processes
  - Relation between the auto-correlation function and the PSD
  - Relation between the cross-correlation function and the xPSD



$\boxed{X(j2\pi f)} \equiv \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$

- Def. Given $x : \mathbb{R} \to \mathbb{C}$, its Fourier transform $X(f)$ is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt. \tag{9.3-1}$$

$\sum_i |x(t_i)|^2 \Delta_i$

$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$= \int_{-\infty}^{\infty} |X(f)|^2 df \simeq \sum_i |X(f_i)|^2 \Delta_i$

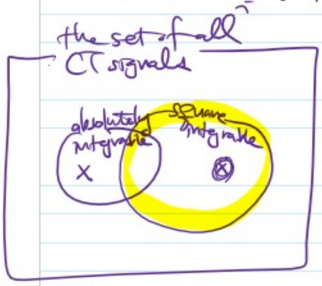
Parseval's relation

- Def. Given $X : \mathbb{R} \to \mathbb{C}$, its inverse Fourier transform $\hat{x}(t)$ is defined as

$$\hat{x}(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df. \tag{9.3-2}$$

$\lim_{\Delta \to \infty} \int_{-\Delta}^{\Delta} x(t) e^{-j2\pi f t} dt$

  - Existence
    - the Lebesgue integral
    - Given a CT signal $x(t)$, the existence of its CTFT is not guaranteed.
    - (a sufficient condition) If $x(t)$ is absolutely integrable, then $X(f)$ exists.
    - Even if $X(f)$ exists, its inverse transform may not exist.



  - Uniqueness
    - If X_1 and X_2 are the FPT of $x(t)$, then they are equal almost everywhere.
    - If the IFT of $X(f)$ exists, then it is unique in the sense of a. e. equality.
- Def. Given $x : \mathbb{R} \to \mathbb{C}$, its Fourier-Plancherel transform $X(f)$ is defined as

$$X(f) = \lim_{\Delta \to \infty} \int_{-\Delta}^{\Delta} x(t) e^{-j2\pi f t} dt, \tag{9.3-3}$$

where the limit is in the MS sense.

- Def. Given $X : \mathbb{R} \to \mathbb{C}$, its inverse Fourier-Plancherel transform $\hat{x}(t)$ is defined as

$$\hat{x}(t) = \lim_{\Delta \to \infty} \int_{-\Delta}^{\Delta} X(f) e^{j2\pi f t} df. \tag{9.3-4}$$

  – Existence
    * the Lebesgue integral
    * Given a CT signal $x(t)$, the existence of its CT FPT is not guaranteed.
    * (a sufficient condition) If $x(t)$ is square integrable, then $X(f)$ exists and square integrable.
  – Uniqueness
    * If X_1 and X_2 are the FPT of $x(t)$, then they are equal almost everywhere.
    * If the IFT of $X(f)$ exists, then it is unique.
- The spectral properties of a finite-energy deterministic signal $x(t)$ are characterized by its FT.
  – The function $X(f)$ is sometimes simply called the spectrum of $x(t)$.
  – It describes how the signal resides in the frequency domain.
  – $x(t)$ can be recovered from $X(f)$ by means of the CT inverse Fourier transform defined as

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df. \tag{9.3-5}$$

  – If $x(t)$ is a finite-energy signal, then
  – $|X(f)|^2$ is called the energy spectral density or energy density spectrum of $x(t)$.
- $X(f)$ may not exist for most sample functions of a random process.
  – The Fourier transform exists for absolutely integrable functions.
  – The Fourier-Plancherel transform exists for square integrable functions.
  – A sample function may be neither absolutely integrable nor square integrable.
  – A sample function may be a power signal.
  – the power spectral density (PSD) or power density spectrum of a random process
- Energy Spectral Density (ESD) of a Finite-Energy Deterministic CT Signal
  – Definition. The energy of a deterministic CT signal $x(t)$ is defined as

$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t.$$

    * The energy is a non-negative real number.
  – Definition. If $0 < E_{xx} < \infty$, then $x(t)$ is called a finite-energy signal or an energy signal.
  – Definition. When $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(f)$, the energy spectral density (ESD) of an energy signal $x(t)$ is defined as

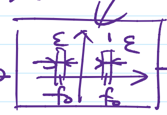
$$|X(f)|^2.$$

$|Y_{f_0,\varepsilon}(f)|^2 = |X(f)|^2 |H_{f_0,\varepsilon}(f)|^2$

$\approx |X(f_0)|^2 \varepsilon$

real finite-energy signal

h(t)



x(t)

$y_{f_0,\varepsilon}(t) = x(t) * h_{f_0,\varepsilon}(t)$

$E_{y_{f_0,\varepsilon}} = \int_{-\infty}^{\infty} |Y_{f_0,\varepsilon}(f)|^2 df$

$\approx |X(f_0)|^2 2\varepsilon.$

    * The ESD is a non-negative real function of f. Moreover, it is symmetric if $x(t)$ is real.

∗ The integration of the ESD leads to E_{xx} because

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f.$$

∗ Fig. Bandpass filtering of $x(t)$ at f_0 with bandwidth ϵ

$$\begin{aligned}\int_{-\infty}^{\infty} |y(t)|^2 \,\mathrm{d}t &= \int_{-\infty}^{\infty} |Y(f)|^2 \,\mathrm{d}f \\ &= \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 \,\mathrm{d}f \\ &\cong \epsilon |X(f_0)|^2\end{aligned}$$

by Parseval's theorem.

∗ In general, the ESD does NOT fully characterize the signal. It partly characterizes the signal in the FD because $x(t)$ cannot be recovered from $|X(f)|^2$.

#

• Instantaneous Power vs. Average Power

$\lim_{\Delta\to\infty} \frac{1}{1} \int_{-\Delta}^{\Delta} |x(t)|^2 dt$

– Energy revisited:

$$\boxed{E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t}$$

$$P_{xx} \triangleq \boxed{\lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} |x(t)|^2 dt} = \langle |x(t)|^2 \rangle$$

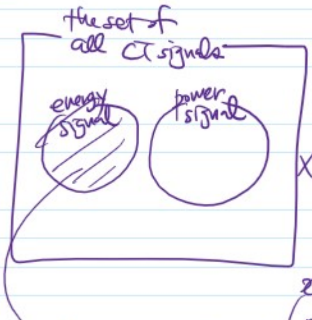
average power

∗ power = energy per unit time

∗ energy = power times time

∗ Thus, $|x(t)|^2$ is called the instantaneous power of $x(t)$.

∗ The instantaneous power is a non-negative real number.



$X(f) = \frac{1}{2}(\delta(f-f_0) + \delta(f+f_0))$

$x(t) = \cos(2\pi f_0 t)$

$|x(t)|^2$ $\quad$ $|x(t)|^2$ $\quad$ $P_{xx} = \frac{1}{2}$

$|X(f)|^2$ $\quad$? $\quad$ $S_{xx}(f) = ?$

$\frac{1}{4}\delta(f-f_0) + \frac{1}{4}\delta(f+f_0)$

Def. The time average of a deterministic CT signal $x(t)$ is defined by

$$\overline{x} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta+c}^{\Delta+c} x(t)\, dt \qquad \langle x(t) \rangle \qquad A[x(t)]$$

Example 1. Given $x(t) = x(t+T)$, $\forall t$,

$$\overline{x} = \frac{1}{T}\int_{\langle T \rangle} x(t)\, dt$$

Example 1. Given $x(t) = x(t+T),\ \forall t$,

$$\bar{x} = \frac{1}{T}\int_{\langle T\rangle} x(t)\,dt$$

It involves no limit operation

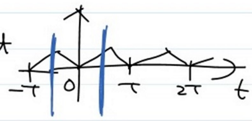
If $\bar{x}$ exists, then

$$\bar{x} = \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} x(t)\,dt$$

$$= \lim_{N\to\infty} \frac{1}{NT}\int_{-\frac{TN}{2}}^{\frac{NT}{2}} x(t)\,dt$$

$$= \lim_{N\to\infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt$$

$$= \frac{1}{T}\int_{\langle T\rangle} x(t)\,dt$$



Example 2.

Given $x(t) = \begin{cases} x(t), & -\frac{T}{2} \le t < \frac{T}{2} \\ 0, & \text{elsewhere,} \end{cases}$

$$\bar{x} = \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} x(t)\,dt \qquad \forall\ \Delta \ge \frac{T}{2}$$

$$= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\underbrace{\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt}_{\text{If finite, then}}$$

$$= 0.$$

$\Rightarrow$ If $\int_{-\infty}^{\infty} x(t)\,dt < \infty$, then $\bar{x} = 0$

– Definition. The average power of a deterministic CT signal $x(t)$ is defined as

$$P_{xx} \triangleq \mathbb{A}\{|x(t)|^2\} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} |x(t)|^2 \mathrm{d}t.$$

where $\mathbb{A}\{\cdot\}$ denotes the time-average.

∗ The average power is a non-negative real number.

– Definition. If $0 < P_{xx} < \infty$, then $x(t)$ is called a finite-power signal or a power signal.

– Note that

$$0 < P_{xx} < \infty \implies E_{xx} = \infty$$
$$0 < E_{xx} < \infty \implies P_{xx} = 0$$

• Power Spectral Density (PSD) of a Finite-Power Deterministic CT Signal

– Definition. When

$$x_\Delta(t) = \begin{cases} x(t), & -\Delta < t < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

and $x_\Delta(t) \overset{\mathcal{F}}{\leftrightarrow} X_\Delta(f)$, the power spectral density (PSD) of a power signal $x(t)$ is defined as

$$S_{xx}(f) \triangleq \lim_{\Delta\to\infty} \frac{|X_\Delta(f)|^2}{2\Delta}.$$

$X_\Delta(f) \triangleq \int_{-\Delta}^{\Delta} x(t) e^{-j2\pi ft} dt$

∗ The PSD is a non-negative real function of f. If $x(t)$ is real, then it is symmetrical.

$|X_\Delta(f)|^2$

$\lim_{\Delta\to\infty} \frac{\int |x(t)|^2 dt}{2\Delta}$ = Joules/sec = Watts

$x(t) = \cos(2\pi f_0 t)$

$S_{xx}(f) = \frac{1}{4}\left(\delta(f-f_0) + \delta(f+f_0)\right)$

$x(t) = \sum_{\ell=1}^{L} a_\ell e^{j2\pi f_\ell t} \longrightarrow S_{xx}(f) = \sum_{\ell=1}^{L} |a_\ell|^2 \delta(f-f_\ell)$

∗ Note that the PSD is the limit of the ESD $|X_\Delta(f)|^2$ divided by time 2Δ.

∗ The integration of the PSD leads to P_{xx} because

$$\begin{aligned} P_{xx} &= \lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{|x(t)|^2}{2\Delta}\mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{|x_\Delta(t)|^2}{2\Delta}\mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{|X_\Delta(f)|^2}{2\Delta}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} \lim_{\Delta\to\infty} \frac{|X_\Delta(f)|^2}{2\Delta}\mathrm{d}f \\ &= \int_{-\infty}^{\infty} S_{xx}(f)\mathrm{d}f \quad \text{(if the limit exists)} \end{aligned}$$

$$= \int_{-\infty} S_{xx}(f)\mathrm{d}f \quad \text{(if the limit exists)}$$

* In general, the PSD does NOT fully characterize the signal. It partly characterizes the signal in the FD because $x(t)$ cannot be recovered from $S_{xx}(f)$.

- Cross-Power Spectral Density (xPSD) of Two Finite-Power Deterministic CT Signals
  - Q. Given two finite-power deterministic CT power signals $x(t)$ and $y(t)$, what is the PSD of $w(t) = x(t) + y(t)$?
  - A.
    * In general, $P_{ww} \neq P_{xx} + P_{yy}$.
      · Definition. The cross power between power signals $x(t)$ and $y(t)$ is defined as
      $$P_{xy} \triangleq \mathbb{A}\{x(t)y^*(t)\}.$$
      · In general, the cross power is complex-valued. If $x(t)$ and $y(t)$ are real, then it is a real number that is positive, negative, or zero.

      · Thus, $P_{ww} = P_{xx} + P_{xy} + P_{yx} + P_{yy} = P_{xx} + 2\,\mathrm{Re}\{P_{xy}\} + P_{yy}$.

* Similarly, $S_{ww}(f) \neq S_{xx}(f) + S_{yy}(f)$ in general.
  · Definition. The cross-power spectral density (xPSD) between power signals $x(t)$ and $y(t)$ is defined as
  $$S_{xy}(f) \triangleq \lim_{\Delta\to\infty} \frac{X_\Delta(f)Y_\Delta^*(f)}{2\Delta}.$$
  · In general, the xPSD is a complex-valued function. If $x(t)$ and $y(t)$ are real, then it is a conjugate symmetrical function.

* Thus, $S_{ww}(f) = S_{xx}(f) + S_{xy}(f) + S_{yx}(f) + S_{yy}(f) = S_{xx}(f) + 2\,\mathrm{Re}\{S_{xy}(f)\} + S_{yy}(f)$.

- Time Average of a CT Random Process
  - Q. Given a CT random process $X(t)$, what is the time average $\mathbb{A}\{X(t)\}$ of it?
  - A. It is a random variable because

$$\mathbb{A}\{X(t)\} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} X(t)\mathrm{d}t$$

    is a scalar with probabilistic description.
    * cf. The ensemble average or statistical average $\mathbb{E}\{X(t)\}$ is a deterministic CT signal.

    * Example. Given $\Pr(X(t) = \cos(2\pi t)) = \Pr(X(t) = 0) = \frac{1}{2}$, find $\mathbb{E}\{|X(t)|^2\}$ and $\mathbb{A}\{|X(t)|^2\}$.

$$\mathbb{E}\{|X(t)|^2\} = \frac{1}{2}\cos^2(2\pi t)$$

      and

$$\Pr(\mathbb{A}\{|X(t)|^2\} = 1/2) = \Pr(\mathbb{A}\{|X(t)|^2\} = 0) = 1/2.$$

- PSD of a CT Random Process
  - Definition. The average power of a CT random process $X(t)$ is defined as

$$P_{XX} \triangleq \mathbb{A}[\mathbb{E}\{|X(t)|^2\}].$$

    * This definition is backward compatible with that for a deterministic CT signal $x(t)$.
    * It is the time average of the ensemble average of $|X(t)|^2$.
    * $P_{XX} = \mathbb{E}[\mathbb{A}\{|X(t)|^2\}]$.
    * Thus, it is a deterministic non-negative real number.

    * Example.

– Definition. When

$$X_\Delta(t) = \begin{cases} X(t), & -\Delta < t < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

and $X_\Delta(t) \overset{\mathcal{F}}{\leftrightarrow} X_\Delta(f)$, the PSD of a CT random process $X(t)$ is defined as

$$S_{XX}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\big[|X_\Delta(f)|^2\big]}{2\Delta}$$

* This definition is backward compatible with that for a deterministic CT signal $x(t)$.
* The PSD is a non-negative real function of f. If $X(t)$ is real, then it is symmetrical.
* Note that the PSD is the limit of $\mathbb{E}\{|X_\Delta(f)|^2\}$ divided by time 2Δ.

* The integration of the PSD leads to P_{XX} because

$$\begin{aligned} P_{XX} &\triangleq \mathbb{A}\big[\mathbb{E}\big[|X(t)|^2\big]\big] \\ &= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} \mathbb{E}\big[|X(t)|^2\big]\,\mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{\mathbb{E}\big[|X_\Delta(t)|^2\big]}{2\Delta}\,\mathrm{d}t \\ &= \lim_{\Delta\to\infty} \mathbb{E}\left[\int_{-\infty}^{\infty} \frac{|X_\Delta(f)|^2}{2\Delta}\,\mathrm{d}f\right] \\ &= \int_{-\infty}^{\infty} \lim_{\Delta\to\infty} \frac{\mathbb{E}\big[|X_\Delta(f)|^2\big]}{2\Delta}\,\mathrm{d}f \\ &= \int_{-\infty}^{\infty} S_{XX}(f)\,\mathrm{d}f \end{aligned}$$

* In general, the PSD does NOT fully characterize the CT random process. It partly characterizes the signal in the FD.

- xPSD of Two CT Random Processes
  - Q. Given two CT random processes $X(t)$ and $Y(t)$, what is the PSD of $W(t) = X(t) + Y(t)$?
  - A.
    * In general, $P_{WW} \neq P_{XX} + P_{YY}$.
      · Definition. The cross power between CT random processes $X(t)$ and $Y(t)$ is defined as
      $$P_{XY} \triangleq \mathbb{A}[\mathbb{E}\{X(t)Y^*(t)\}].$$
      · This definition is backward compatible with that for deterministic CT signals $x(t)$ and $y(t)$.
      · In general, the cross power is complex-valued. If $X(t)$ and $Y(t)$ are real, then it is a real number that is positive, negative, or zero.
      · Thus, $P_{WW} = P_{XX} + P_{XY} + P_{YX} + P_{YY} = P_{XX} + 2\,\mathrm{Re}\{P_{XY}\} + P_{YY}$.
    * Similarly, $S_{WW}(f) \neq S_{XX}(f) + S_{YY}(f)$ in general.
      · Definition. The xPSD between two CT random processes $X(t)$ and $Y(t)$ is defined as
      $$S_{XY}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\{X_\Delta(f)Y_\Delta^*(f)\}}{2\Delta}.$$
      · In general, the xPSD is a complex-valued function. If $X(t)$ and $Y(t)$ are real, then it is a conjugate symmetrical function.

    * Thus, $S_{WW}(f) = S_{XX}(f) + S_{XY}(f) + S_{YX}(f) + S_{YY}(f) = S_{XX}(f) + 2\,\mathrm{Re}\{S_{XY}(f)\} + S_{YY}(f)$.

- Q. Why is the PSD so important in wireless communications, though it partially characterizes a random process?
- A.
  - The signal transmitted by a Tx is well modeled by a CT real-valued bandpass random process.
  - Regulatory organizations impose restrictions on the peak power, the average power, the PSD, etc. of the transmitted signal.
  - spectral mask
  - Thus, it is crucial to design standards and systems that do not violate such restrictions.
- Q. Recall that the PSD of a CT random process $X(t)$ is defined as
$$S_{XX}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\left[|X_\Delta(f)|^2\right]}{2\Delta}.$$
Is there any alternative form with which we can easily compute the PSD?
  - Notice that the PSD is related only to the second-order moment of $X(t)$.

Thm. Given a deterministic power signal $x(t)$, the PSD $S_{xx}(f)$ can be alternatively found as

$$S_{xx}(f) = \mathcal{F}_\tau\left\{\mathbb{A}_t\left[x(t+\tau)x^*(t)\right]\right\}$$

$$\lim_{\Delta\to\infty}\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} x(t+\tau)x^*(t)\,dt$$

$$= R_{xx}(\tau)$$

$$\int_{-\infty}^{\infty} R_{xx}(\tau)\, e^{-j2\pi f\tau}\, d\tau$$

$$S_{xx}(f) \triangleq \lim_{\Delta\to\infty} \frac{|X_\Delta(f)|^2}{2\Delta}$$

  - The 2nd-order TD characterization is done by the auto-correlation function defined as
  $$R_{XX}(t+\tau, t) \triangleq \mathbb{E}\{X(t+\tau)X^*(t)\}.$$
  - The question is to find the relation between $R_{XX}(t+\tau, t)$ and $S_{XX}(f)$.

Def.

$$S_{XX}(f) = \lim_{\Delta\to\infty} \frac{1}{2\Delta}\, \mathbb{E}\{|X_\Delta(f)|^2\}$$

Thm.

$$S_{XX}(f) = \mathcal{F}_\tau\left(\mathbb{A}_t\left\{\mathbb{E}[X(t+\tau)X^*(t)]\right\}\right)$$

- A. The PSD can be found from the auto-correlation function as

$$S_{XX}(f) = \mathcal{F}\{\mathbb{A}[R_{XX}(t+\tau, t)]\}\,.$$

  - $S_{XX}(f) = \mathcal{F}\{\mathbb{A}[\mathbb{E}\{X(t+\tau)X^*(t)\}]\}$
  - The PSD is NOT affected by the complementary/pseudo auto-correlation function $\tilde{R}_{XX}(t+\tau, t)$.
  - $\mathbb{A}[R_{XX}(t+\tau, t)]$ is a function only of τ and often denoted by $\mathcal{R}_{XX}(\tau)$.
  - $\mathcal{R}_{XX}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XX}(f)$
- (Wiener-Khinchin Theorem) If $X(t)$ is WSS, then

$$R_{XX}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XX}(f),$$

i.e.,

$$\begin{cases} S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\ R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)e^{\mathrm{j}2\pi f\tau}\,\mathrm{d}f. \end{cases}$$

  - Example. Find the PSD of

$$\begin{aligned} X(t) = \cos(2\pi f_0 t) &= \frac{e^{\mathrm{j}2\pi f_0 t} + e^{-\mathrm{j}2\pi f_0 t}}{2}, \forall t \\ \overset{\mathcal{F}}{\longleftrightarrow} \quad & \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0). \end{aligned}$$

    * $X(t)$ is deterministic, so that no statistical average is required.
    * Since $R_{XX}(t+\tau, t)$ is given by

$$\begin{aligned} R_{XX}(t+\tau, t) &= \cos 2\pi f_0(t+\tau)\cos 2\pi f_0(t) \\ &= \frac{1}{2}\left\{\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0\tau)\right\}, \end{aligned}$$

    $\mathcal{R}_{XX}(\tau)$ is obtained as

$$\mathbb{A}[R_{XX}(t+\tau, t)] = \frac{1}{2}\cos(2\pi f_0\tau).$$

    Thus,

$$S_{XX}(f) = \frac{1}{4}\delta(f - f_0) + \frac{1}{4}\delta(f + f_0).$$

– (Generalization) The PSD of $X(t) = \sum_{i \in I} a_i e^{j2\pi f_i t}$ is given by

$$S_{XX}(f) = \sum_{i \in I} |a_i|^2 \delta(f - f_i).$$

∗ Fig. Graphs of $X(f)$ and $S_{XX}(f)$

∗ The PSD is NOT a full characterization of the random process. Later, we will see that it is the case when $X(t)$ is a real-valued WSS Gaussian random process.

- Q. Recall that the xPSD of two CT random processes $X(t)$ and $Y(t)$ is defined as

$$S_{XY}(f) \triangleq \lim_{\Delta \to \infty} \frac{\mathbb{E}\{X_\Delta(f) Y_\Delta^*(f)\}}{2\Delta}.$$

Is there any alternative form with which we can easily compute the xPSD?

- A. The xPSD can be found from the cross-correlation function as

$$S_{XY}(f) = \mathcal{F}\{\mathbb{A}[R_{XY}(t+\tau, t)]\}.$$

– $S_{XY}(f) = \mathcal{F}\{\mathbb{A}[\mathbb{E}\{X(t+\tau)Y^*(t)\}]\}$

– The PSD is NOT affected by the complementary/pseudo cross-correlation function $\tilde{R}_{XY}(t+\tau, t)$.

– $\mathbb{A}[R_{XY}(t+\tau, t)]$ is a function only of τ and often denoted by $\mathcal{R}_{XY}(\tau)$.

– $\mathcal{R}_{XY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XY}(f)$

– If $X(t)$ and $Y(t)$ are jointly WSS, then

$$R_{XY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XY}(f),$$

- Proof.

$$
\begin{aligned}
S_{XX}(f) &= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[|X_\Delta(f)|^2\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[X_\Delta(f)\,(X_\Delta(f))^*\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[\left(\int_{-\infty}^{\infty} X_\Delta(t_1)e^{-j2\pi f t_1}\,\mathrm{d}t_1\right)\left(\int_{-\infty}^{\infty} X_\Delta(t_2)e^{-j2\pi f t_2}\,\mathrm{d}t_2\right)^*\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta} \mathbb{E}[X(t_1)X^*(t_2)]\,e^{-j2\pi f(t_1-t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\int_{-\Delta-t}^{\Delta-t} \mathbb{E}[X(t+\tau)X^*(t)]\,e^{-j2\pi f\tau}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta-t}^{\Delta-t}\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)e^{-j2\pi f\tau}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \lim_{\Delta\to\infty} \int_{-\Delta-t}^{\Delta-t}\frac{1}{2\Delta}\left(\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)\,\mathrm{d}t\right)e^{-j2\pi f\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty}\left(\lim_{\Delta\to\infty}\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)\,\mathrm{d}t\right)e^{-j2\pi f\tau}\,\mathrm{d}\tau \\
&= \mathcal{F}\{\mathbb{A}[R_{XX}(t+\tau,t)]\}
\end{aligned}
$$

- Properties of $S_{XX}(f)$ and $S_{XY}(f)$
  - Use either the definition or the theorem.
  - (real, non-negative PSD) $S_{XX}(f) \geq 0, \forall f$.
  - $\dot{X}(t) \triangleq \frac{\mathrm{d}}{\mathrm{d}t}X(t)$
    $S_{\dot{X}\dot{X}}(f) = (2\pi f)^2 S_{XX}(f)$
  - (conjugate pair) $S_{XY}(f) = S^*_{YX}(f), \forall f$.
    - $\mathrm{Re}\{S_{XY}(f)\} = \mathrm{Re}\{S_{YX}(f)\}$ and $\mathrm{Im}\{S_{XY}(f)\} = -\,\mathrm{Im}\{S_{YX}(f)\}$.
  - If $X(t)$ and $Y(t)$ are orthogonal, i.e., $R_{XY}(t+\tau,t) = 0, \forall t, \forall \tau$, then $S_{XY}(f) = 0, \forall f$. However, the converse is not true in general.

– If $X(t)$ and $Y(t)$ are jointly WSS, then
  * $R_{XX}(\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} S_{XX}(f)$
  * $R_{YY}(\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} S_{YY}(f)$
  * $R_{XY}(\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} S_{XY}(f)$
  * $R_{YX}(\tau) = R^*_{XY}(-\tau) \stackrel{\mathcal{F}}{\longleftrightarrow} S_{YX}(f) = S^*_{XY}(f)$
  * $P_{XX} = R_{XX}(0), P_{YY} = R_{YY}(0), P_{XY} = R_{XY}(0) = R^*_{YX}(0) = P^*_{YX}$

– If $X(t)$ and $Y(t)$ are real-valued, then
  * (real, non-negative, symmetric PSD) $S_{XX}(f) = S_{XX}(-f), \forall f$

  * (conjugate symmetrical xPSD) $S_{XY}(f) = S^*_{XY}(-f), \forall f$

• Two most important CT random processes

– $\begin{cases} \text{continuous random process: CT Gaussian random process} \\ \text{discrete random process: Poisson random process} \end{cases}$

Fig.