Chapter 9
GENERAL CONCEPTS

Handout by Joon Ho Cho
based on A. Papoulis & S. U. Pillai, *Probability, Random Variables and Stochastic Processes, 4th ed.*

EECE 574, Spring 2023

9.1 Definitions

9.5 APPENDIX 9A Continuity, Differentiation, Integration

9.2 Systems with Stochastic Inputs

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9.4 Spectral Characteristics of System Response

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9.0 Introduction

9.5 Continuity, Differentiation, Integration

- Q. What is

$$\lim_{t \to t_0} X(t)$$

  ?

  – Since a RP is a collection of RVs parameterized by t, it means $\lim_{t \to t_0} X(t, s)$.
  – If the limit exists in some sense, then it must be a RV.
  – We need to use the notion of the convergence of a sequence of RVs to a RV.

- What do we mean by the continuity

$$\lim_{t \to t_0} X(t) = X(t_0)$$

  ?

  – a.s.
  – in the pth mean
  – in probability
  – in distribution

- We mainly consider the notion of the limit in the mean-square sense, i.e., we are interested in the case with

$$\text{l.i.m.}_{t \to t_0} X(t) = X(t_0).$$

Mean-Square Calculus

- From Chapter 9 of Stark and Woods, *Probability, Random Processes, and Estimation Theory for Engineers*, 2nd ed.
- Def. A random process $X(t)$ is continuous in the mean-square sense at the point t if

$$E[|X(t+\epsilon) - X(t)|^2] \to 0, \text{ as } \epsilon \to 0.$$

- Def. A random process $X(t)$ is mean-square (m.-s.) continuous if $X(t)$ is continuous in the mean-square sense for all t.
  - Theorem. The random process $X(t)$ is m.-s. continuous at t iff $R_{XX}(t_1, t_2)$ is continuous at the point $t_1 = t_2 = t$.

    proof.

    $$E[|X(t+\epsilon)-X(t)|^2] = R_{XX}(t+\epsilon, t+\epsilon)-R_{XX}(t, t+\epsilon)-R_{XX}(t+\epsilon, t)+R_{XX}(t, t)$$

    Clearly the right-hand side goes to zero as $\epsilon \to 0$ if the deterministic function R_{XX} is continuous at $t_1 = t_2 = t$. The only if part is somewhat complicated and will be omitted.
  - Corollary. A WSS random process is m.-s. continuous for all t iff $R_{XX}(\tau)$ is continuous at $t = 0$.

- Def. The random process $X(t)$ has a mean-square derivative at t if the mean-square limit of $[X(t+\epsilon) - X(t)]/\epsilon$ exists as $\epsilon \to 0$.
  - Theorem. A random process $X(t)$ with autocorrelation function $R_{XX}(t_1, t_2)$ has a m.-s. derivative at time t iff $\partial^2 R_{XX}(t_1, t_2)/\partial t_1 \partial t_2$ exists at $t_1 = t_2 = t$.
  - Theorem. If a random process $X(t)$ with mean function $\mu_X(t)$ and correlation function $R_{XX}(t_1, t_2)$ has an m.-s. derivative $X'(t)$, then the mean and correlation function of $X'(t)$ are given by

$$\mu_{X'}(t) = d\mu_X(t)/dt$$

    and

$$R_{X'X'}(t_1, t_2) = \partial^2 R_{XX}(t_1, t_2)/\partial t_1 \partial t_2.$$

  - Theorem. The m.-s. derivative of a WSS random process $X(t)$ exists at t if the autocorrelation function $R_{XX}(\tau)$ has derivatives up to order two at $\tau = 0$.

- Def. The mean-square integral of $X(t)$ over the interval (T_1, T_2) is denoted I. It exists when

$$\lim_{n\to\infty} E\left[\left|I - \sum_{n=1}^{N} X(t_i)\triangle t_i\right|^2\right] = 0.$$

We give I the following symbol:

$$I \triangleq \int_{T_1}^{T_2} X(t)dt.$$

- Def. The weighted mean-square integral of $X(t)$ over the interval (T_1, T_2) is defined by

$$\lim_{n\to\infty} E\left[\left|I - \sum_{n=1}^{N} h(t_i)X(t_i)\triangle t_i\right|^2\right] = 0,$$

when the limit exists. We give it the following symbol:

$$I \triangleq \int_{T_1}^{T_2} h(t)X(t)dt.$$

9.2 Systems with Stochastic Inputs

- a system = an operation on $(x(t))_t$ to cause $(y(t))_t$

$$y(t) = M[x(t)]$$

- In this course, only a deterministic system is considered. No random system is considered.
- a memoryless system = an operation on $x(t)$ to cause $y(t)$
- time-invariant system

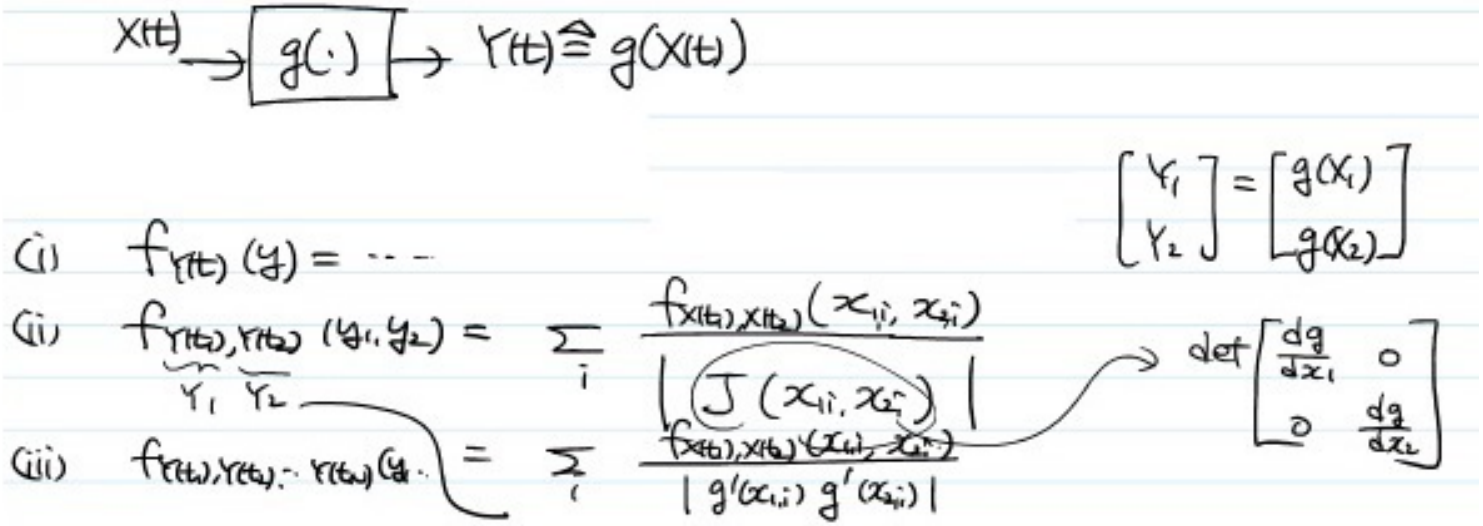
$$y(t - \Delta) = M[x(t - \Delta)], \forall \Delta$$

- memoryless time-invariant system

$$y(t) = g(x(t))$$

where $g(x)$ is a real-valued function

– How to find the distribution of output?



– What are the expectation and moments? Is there a simple way?

- $\mu_{Y(t)} \triangleq E\{Y(t)\} = \int_{-\infty}^{\infty} y\, f_{Y(t)}(y)\, dy$

  $= E\{g(X(t))\} = \int_{-\infty}^{\infty} g(x) f_{X(t)}(x)\, dx.$

- $R_{XX}(t_1, t_2)$

  $R_{XY}(t_1, t_2) = E\{X(t_1)\, Y(t_2)\} = E[\underbrace{X(t_1)}_{x_1}\, g(\underbrace{X(t_2)}_{x_2})]$

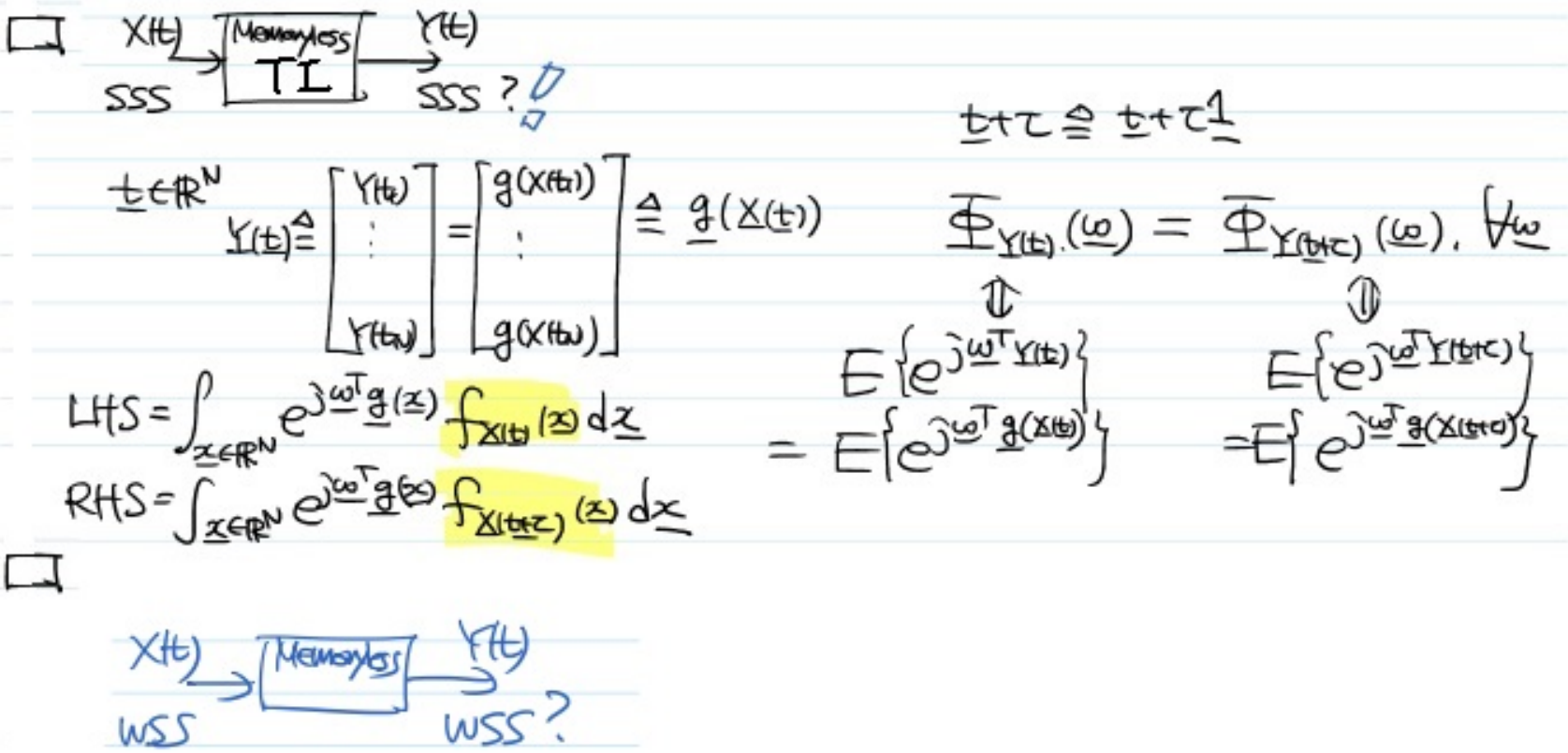
  $= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 g(x_2)\, f_{X(t_1)X(t_2)}(x_1, x_2)\, dx_1 dx_2$

  $R_{YX}(t_1, t_2)$

  $R_{YY}(t_1, t_2)$

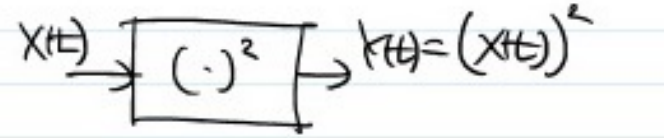
– Q. If the input to a memoryless time-invariant system is SSS, then the output is also SSS?

– Q. If the input to a memoryless time-invariant system is WSS, then the output is also WSS?

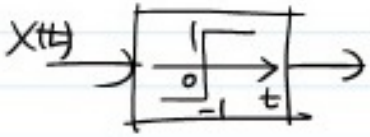


– Examples

– square-law device



– hardlimiter



signum

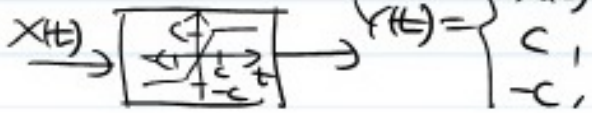
sign

$$E\{Y(t)\} = 1 \times Pr(X(t) > 0) + (-1) \times Pr(X(t) < 0) = 1 - Pr(X(t) \le 0) - Pr(X(t) < 0) = 1 - 2F_{X(t)}(0)$$

MATLAB

$$sign(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

– limiter



Linear Systems

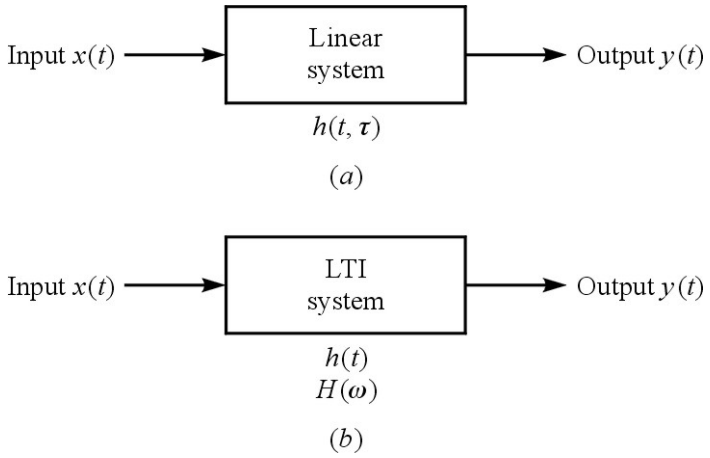


Figure 9.2-1

- Def. A system is linear if $y_n(t) = L[x_n(t)]$ implies

$$y(t) = L\left[\sum_{n=1}^{N} \alpha_n x_n(t)\right] = \sum_{n=1}^{N} \alpha_n y_n(t) \tag{9.2-1}$$

  – the sifting property

$$x(t) = \int_{-\infty}^{\infty} x(\xi)\delta(t-\xi)d\xi \tag{9.2-2}$$

$$y(t) = L[x(t)] = L\left[\int_{-\infty}^{\infty} x(\xi)\delta(t-\xi)d\xi\right] = \int_{-\infty}^{\infty} x(\xi)L[\delta(t-\xi)]d\xi \quad (9.2\text{-}3)$$

– Def. the impulse response of the linear system

$$L[\delta(t-\xi)] = h(t,\xi) \quad (9.2\text{-}4)$$

– the I/O relation of a general linear system

$$y(t) = \int_{-\infty}^{\infty} h(t,\xi)x(\xi)d\xi \quad (9.2\text{-}5)$$

- Def. A linear system is time-invariant

$$h(t,\xi) = h(t-\xi) \quad (9.2\text{-}6)$$

– the I/O relation of an LTI system

$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t-\xi)d\xi \quad (9.2\text{-}7)$$

– the convolution integral of $x(t)$ and $h(t)$

$$y(t) = x(t) * h(t) \quad (9.2\text{-}8)$$

$$y(t) = \int_{-\infty}^{\infty} h(\xi)x(t-\xi)d\xi \tag{9.2-9}$$

– In the time domain, an LTI system is completely characterized by its impulse response.

– In the frequency domain,

$$\begin{aligned} Y(\omega) &= \int_{-\infty}^{\infty} y(t)e^{-\mathrm{j}\omega t}dt = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(\xi)h(t-\xi)d\xi\right]e^{-\mathrm{j}\omega t}dt \\ &= \int_{-\infty}^{\infty} x(\xi)\left[\int_{-\infty}^{\infty} h(t-\xi)e^{-\mathrm{j}\omega(t-\xi)}dt\right]e^{-\mathrm{j}\omega\xi}d\xi \\ &= \int_{-\infty}^{\infty} x(\xi)H(\omega)e^{-\mathrm{j}\omega\xi}d\xi = X(\omega)H(\omega) \end{aligned} \tag{9.2-10}$$

– $H(\omega)$ the transfer function of the system

RANDOM SIGNAL RESPONSE OF LTI SYSTEMS

- $h(t)$ is real.
- temporal characteristic of the random signal response of linear systems

System Response—Convolution

$$y(t) = \int_{-\infty}^{\infty} x(\xi)h(t-\xi)d\xi \tag{9.2-11}$$

$$y(t) = \int_{-\infty}^{\infty} h(\xi)x(t-\xi)d\xi \tag{9.2-12}$$

$$Y(t) = \int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi \tag{9.2-13}$$

Mean and Mean-Squared Value of System Response

- The mean value of $Y(t)$ is

$$\begin{aligned} E[Y(t)] &= E\left[\int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi\right] \\ &= \int_{-\infty}^{\infty} h(\xi)E[X(t-\xi)]d\xi \\ &= h(t) * \mu_X(t). \end{aligned}$$

If $X(t)$ is WSS, then

$$E[Y(t)] = \bar{X}\int_{-\infty}^{\infty} h(\xi)d\xi = \bar{Y}, \qquad \text{a constant.}$$

- The mean-squared value of $Y(t)$ is

$$\begin{aligned} E[Y^2(t)] &= E\left[\int_{-\infty}^{\infty} h(\xi_1)X(t-\xi_1)d\xi_1 \int_{-\infty}^{\infty} h(\xi_2)X(t-\xi_2)d\xi_2\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1)E[X(t-\xi_1)X(t-\xi_2)]h(\xi_2)d\xi_1 d\xi_2 \end{aligned}$$

If $X(t)$ is WSS, then

$$E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\xi_1)R_{XX}(\xi_2-\xi_1)h(\xi_2)d\xi_1 d\xi_2 = \overline{Y^2}, \qquad \text{a constant.}$$

because

$$E[X(t-\xi_1)X(t-\xi_2)] = R_{XX}(\xi_2-\xi_1).$$

Cross-Correlation Functions of Input and Output

- If the system is LTI with $h(t)$ but the input $X(t)$ is not necessarily WSS, then the cross-correlation function of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(t+\tau,t) &= E[X(t+\tau)Y(t)] \\ &= E\left[X(t+\tau)\int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi\right] \\ &= \int_{-\infty}^{\infty} E[X(t+\tau)X(t-\xi)]h(\xi)d\xi. \end{aligned}$$

- If $X(t)$ is WSS, then the cross-correlation function of $X(t)$ and $Y(t)$ is

$$\begin{aligned} R_{XY}(\tau) &= \int_{-\infty}^{\infty} R_{XX}(\tau+\xi)h(\xi)d\xi \\ &= \int_{-\infty}^{\infty} R_{XX}(\tau-\xi)h(-\xi)d\xi \\ &= R_{XX}(\tau) * h(-\tau) \end{aligned}$$

- If the system is LTI with $h(t)$ but the input $X(t)$ is not necessarily WSS, then the cross-correlation function of $Y(t)$ and $X(t)$ is

$$\begin{aligned} R_{YX}(t+\tau,t) &= E[Y(t+\tau)X(t)] \\ &= E\left[\int_{-\infty}^{\infty} h(\xi)Y(t+\tau-\xi)d\xi X(t)\right] \\ &= \int_{-\infty}^{\infty} h(\xi)E[X(t+\tau-\xi)X(t)]d\xi. \end{aligned}$$

- If $X(t)$ is WSS, then the cross-correlation function of $Y(t)$ and $X(t)$ is

$$\begin{aligned} R_{YX}(\tau) &= \int_{-\infty}^{\infty} h(\xi)R_{XX}(\tau-\xi)d\xi \\ &= h(\tau) * R_{XX}(\tau). \end{aligned}$$

Autocorrelation Function of Response

– The autocorrelation function of $Y(t)$ is

$$\begin{aligned} R_{YY}(t+\tau,t) &= E[Y(t+\tau)Y(t)] \\ &= E\left[Y(t+\tau)\int_{-\infty}^{\infty} h(\xi)X(t-\xi)d\xi\right] \\ &= \int_{-\infty}^{\infty} h(\xi_1)E[Y(t+\tau)X(t-\xi)]h(\xi)d\xi. \end{aligned}$$

– If $X(t)$ is WSS, then $Y(t)$ is WSS with

$$\begin{aligned} R_{YY}(t+\tau,t) &= \int_{-\infty}^{\infty} R_{YX}(\tau+\xi)h(\xi)d\xi \\ &= R_{YX}(\tau) * h(-\tau) \\ &= h(\tau) * R_{XX}(\tau) * h(-\tau) \\ &= R_{YY}(\tau) \end{aligned}$$

– The mean-squared value of $Y(t)$ is

$$E[Y^2(t)] = R_{YY}(t,t) = h(t_1) * R_{XX}(t_1,t_2) * h(t_2)|_{t_1=t_2=t}.$$

If $X(t)$ is WSS, then

$$E[Y^2(t)] = R_{YY}(0) = h(\tau) * R_{XX}(\tau) * h(-\tau)|_{\tau=0}.$$

- The relation between autocorrelation function and cross-correlation function is

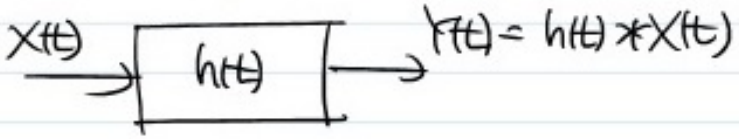
$$\begin{aligned} R_{YY}(t_1, t_2) &= h(t_1) * R_{XX}(t_1, t_2) * h(t_2) \\ &= R_{YX}(t_1, t_2) * h(t_2) \\ &= h(t_1) * R_{XX}(t_1, t_2). \end{aligned}$$

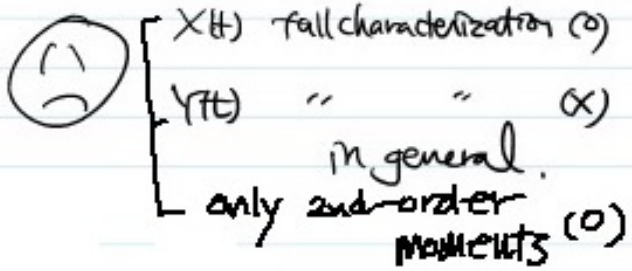
- If $X(t)$ is WSS, then

$$\begin{aligned} R_{YY}(\tau) &= h(\tau) * R_{XX}(\tau) * h(-\tau) \\ &= R_{YX}(\tau) * h(-\tau) \\ &= h(\tau) * R_{XY}(\tau). \end{aligned}$$

- $X(t)$ and $Y(t)$ are jointly WSS.

- If the system is LTI with $h(t)$ but the input is not necessarily WSS, then





$$\mu_{Y(t)} = E\{Y(t)\} = E\{h(t) * X(t)\} = E\left\{\int_{-\infty}^{\infty} h(\tau) X(t-\tau)\, d\tau\right\} = \int_{-\infty}^{\infty} h(\tau)\, E\{X(t-\tau)\}\, d\tau$$

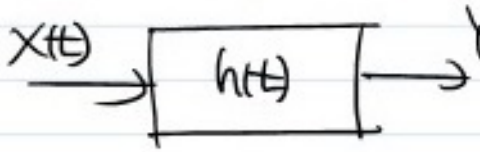
$$= h(t) * \mu_{X(t)}.$$

$$R_{YY}(t_1, t_2) = E\{Y(t_1) Y(t_2)\} = E\left[\{h(t_1) * X(t_1)\}\{h(t_2) * X(t_2)\}\right] = E\left[h(t_1) * \left(X(t_1) X(t_2)\right) * h(t_2)\right]$$

$$= h(t_1) * R_{XX}(t_1, t_2) * h(t_2).$$

$$R_{XY}(t_1, t_2) = R_{XX}(t_1, t_2) * h(t_2) \qquad R_{YX}(t_1, t_2) = h(t_1) * R_{XX}(t_1, t_2).$$

- If the system is LTI with $h(t)$ and the input is WSS, then



If X(t) is WSS, then $\mu_X(t) = \mu_X$, $\forall t$. $R_{XX}(t+\tau, t) = R_{XX}(\tau)$, $\forall \tau \forall t$.

$$\mu_Y(t) = h(t) * \mu_X(t) = \mu_X \int_{-\infty}^{\infty} h(t)\,dt = \mu_X H(0)$$

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) R_{XX}(t_1-\tau_1, t_2-\tau_2) h(\tau_2)\,d\tau_1\,d\tau_2$$

Claim

$$R_{YY}(t+\tau, t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) R_{XX}(t+\tau-\tau_1, t-\tau_2) h(\tau_2)\,d\tau_1\,d\tau_2 = \left(h(\tau) * R_{XX}(\tau)\right) * h(-\tau)$$

$= R_{XX}(\tau - \tau_1 + \tau_2)$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1) R_{XX}(\tau - \tau_1 + \tau_2) h(\tau_2)\,d\tau_1 * h(-\tau)\,d\tau_2$$

$R_{YY}(t+\tau, t) =$ -- ft of τ.

Y(t) is WSS.

- LCCDE

§ 9-2-3 Differential Equations.

$$\sum_{k=1}^{M} a_k y^{(k)}(t) = \sum_{\ell=1}^{N} b_\ell x^{(\ell)}(t)$$

Q1. Is this system LTI?
A1. No. In general, Yes if initial rest (until zero input until zero output) (no zero-input response)

Q2. Any $h(t)$ can be synthesized by an LCCDE?
A2. No, in general. For s.l. M & N, approximately =.

A special case of AR(p)

$$X(t_1)\left(\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_2)\right) = \left(X(t_2)\right) \quad \cdots (*)$$

(i) $\mu_Y(t) =$

$$E\left[\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t)\right] = E\left[X(t)\right]$$

$$E\left[\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{dt^\ell} Y(t)\right] = \sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{dt^\ell} \mu_Y(t) = \mu_X(t)$$

(ii) $R_{YY}(t_1, t_2) =$?

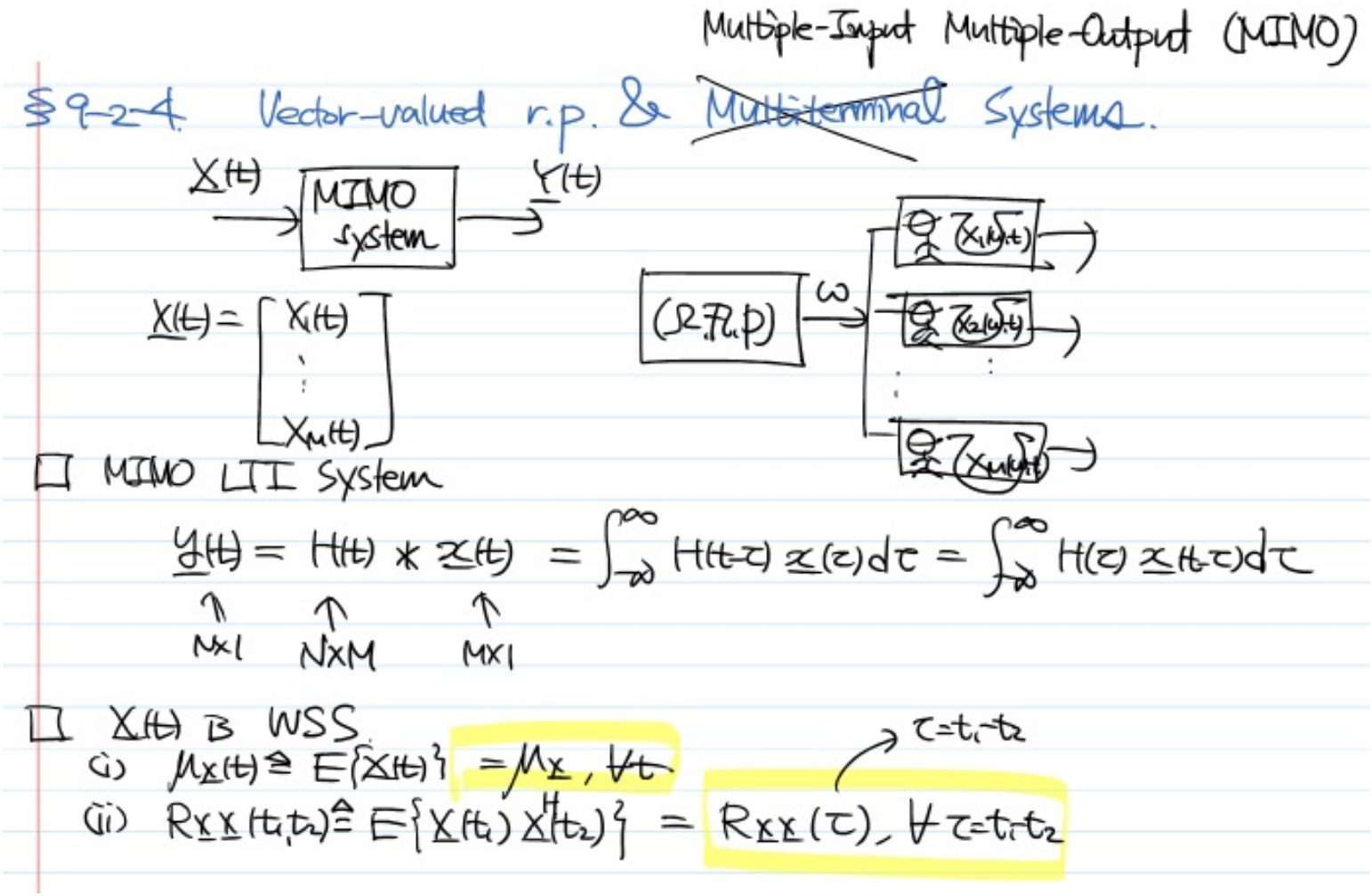
□ $X(t_1)\left(\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_2)\right) = \left(X(t_1)X(t_2)\right)$ ... (*)

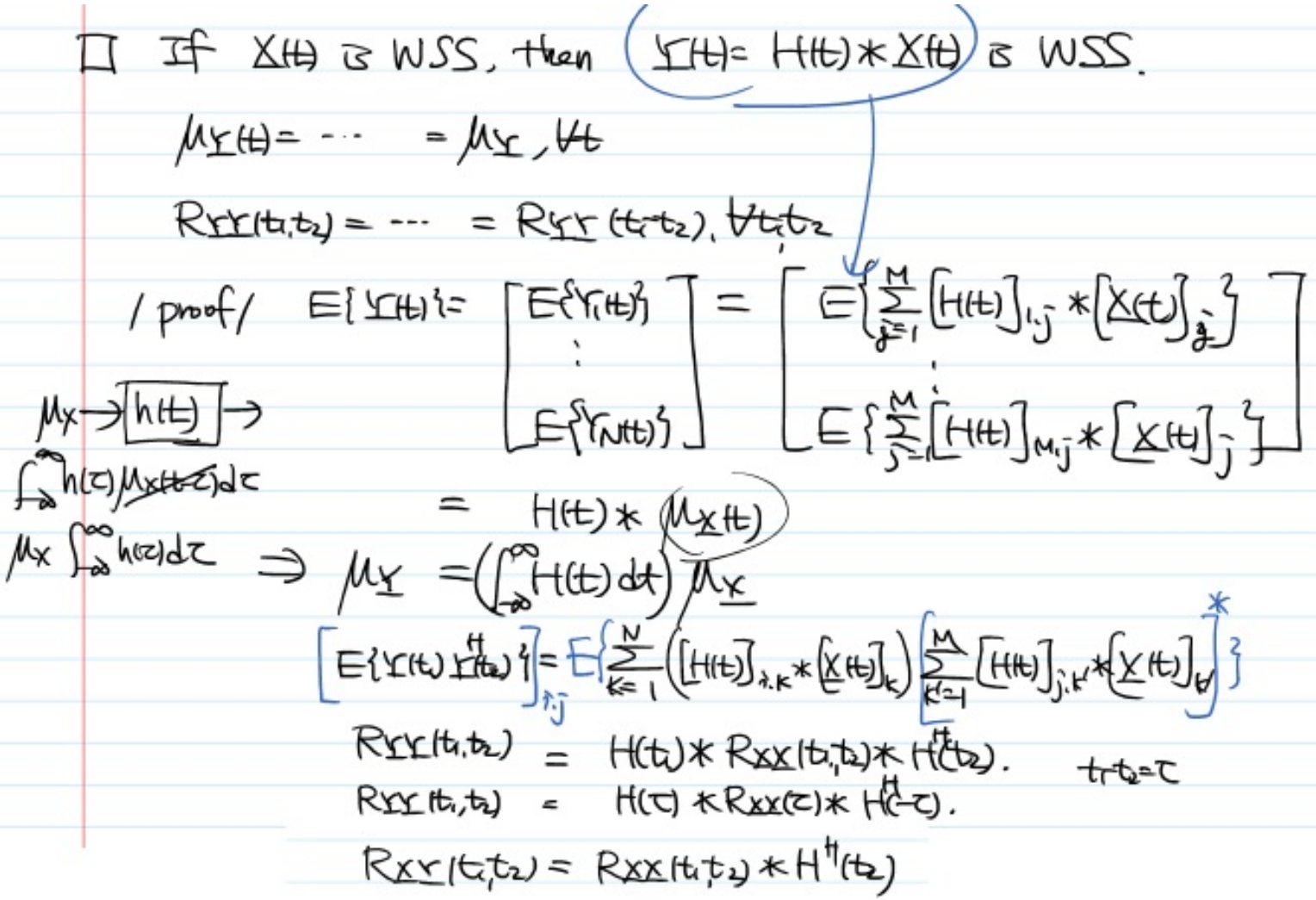
$x(t) \to \boxed{\frac{d}{dt}} \to \frac{dx(t)}{dt} \equiv x(t) \to \boxed{\delta^{(1)}(t)} \to \dot{x}(t)$

(11) $R_{YY}(t_1,t_2) =$

General $X(t)$	WSS $X(t)$
$X(t_1)\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_2) = X(t_1)X(t_2)$	$a_0\mu_Y = \mu_X$.
$\sum_{\ell=0}^{p} a_\ell R_{XY^{(\ell)}}(t_1,t_2) = R_{XX}(t_1,t_2)$	
$Y(t) \to \boxed{\delta^{(\ell)}(t)} \to Y^{(\ell)}(t)$	$\tau = t_1 - t_2$
$\Rightarrow \sum_{\ell=0}^{p} a_\ell \frac{\partial^\ell}{\partial t_1^\ell} R_{XY}(t_1,t_2) = R_{XX}(t_1,t_2)$	$-\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{d\tau^\ell} R_{YY}(\tau) = R_{XX}(\tau)$
Mittal ———	
$\sum_{\ell=0}^{p} a_\ell Y^{(\ell)}(t_1)Y(t_2) = X(t_1)Y(t_2)$	
$\sum_{\ell=0}^{p} a_\ell \frac{\partial^\ell}{\partial t_1^\ell} R_{YY}(t_1,t_2) = R_{XY}(t_1,t_2)$	$\sum_{\ell=0}^{p} a_\ell \frac{d^\ell}{d\tau^\ell} R_{YY}(\tau) = R_{XY}(\tau)$.
Mittal ———	

Vector Processes and Multiterminal Systems




If X(t) is WSS, then Y(t) = H(t) * X(t) is WSS.
μY(t) = ··· = μY, ∀t
RYY(t1,t2) = ··· = RYY(t1-t2), ∀t1,t2
/ proof/
μX → h(t) →
∫ h(τ) μX(t-τ) dτ
μX ∫ h(τ) dτ
= H(t) * μX(t)
⇒ μY = (∫ H(t) dt) μX
RXY(t1,t2) = H(t1) * RXX(t1,t2) * H^H(t2).
t1-t2=τ
RXY(t1,t2) = H(τ) * RXX(τ) * H^H(-τ).
RXY(t1,t2) = RXX(t1,t2) * H^H(t2)

9.3 The Power Spectrum

- Preview.
  - Energy spectral density (ESD) and power spectral density (PSD) of deterministic CT signals
  - PSD of a CT random process
  - cross PSD of two CT random processes
  - Relation between the auto-correlation function and the PSD
  - Relation between the cross-correlation function and the xPSD

- Def. Given $x : \mathbb{R} \to \mathbb{C}$, its Fourier transform $X(f)$ is defined as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-\mathrm{j}2\pi ft} dt. \tag{9.3-1}$$

- Def. Given $X : \mathbb{R} \to \mathbb{C}$, its inverse Fourier transform $\hat{x}(t)$ is defined as

$$\hat{x}(t) = \int_{-\infty}^{\infty} X(f) e^{\mathrm{j}2\pi ft} df. \tag{9.3-2}$$

  - Existence
    - the Lebesgue integral
    - Given a CT signal $x(t)$, the existence of its CTFT is not guaranteed.
    - (a sufficient condition) If $x(t)$ is absolutely integrable, then $X(f)$ exists.
    - Even if $X(f)$ exists, its inverse transform may not exist.
  - Uniqueness
    - If X_1 and X_2 are the FPT of $x(t)$, then they are equal almost everywhere.
    - If the IFT of $X(f)$ exists, then it is unique in the sense of a. e. equality.

- Def. Given $x : \mathbb{R} \to \mathbb{C}$, its Fourier-Plancherel transform $X(f)$ is defined as

$$X(f) = \lim_{\Delta \to \infty} \int_{-\Delta}^{\Delta} x(t) e^{-\mathrm{j}2\pi ft} dt, \tag{9.3-3}$$

  where the limit is in the MS sense.

- Def. Given $X : \mathbb{R} \to \mathbb{C}$, its inverse Fourier-Plancherel transform $\hat{x}(t)$ is defined as

$$\hat{x}(t) = \lim_{\Delta \to \infty} \int_{-\Delta}^{\Delta} X(f) e^{\mathrm{j}2\pi f t} df. \tag{9.3-4}$$

  - Existence
    - the Lebesgue integral
    - Given a CT signal $x(t)$, the existence of its CT FPT is not guaranteed.
    - (a sufficient condition) If $x(t)$ is square integrable, then $X(f)$ exists and square integrable.
  - Uniqueness
    - If X_1 and X_2 are the FPT of $x(t)$, then they are equal almost everywhere.
    - If the IFT of $X(f)$ exists, then it is unique.

- The spectral properties of a finite-energy deterministic signal $x(t)$ are characterized by its FT.
  - The function $X(f)$ is sometimes simply called the spectrum of $x(t)$.
  - It describes how the signal resides in the frequency domain.
  - $x(t)$ can be recovered from $X(f)$ by means of the CT inverse Fourier transform defined as

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{\mathrm{j}2\pi f t} df. \tag{9.3-5}$$

  - If $x(t)$ is a finite-energy signal, then
  - $|X(f)|^2$ is called the energy spectral density or energy density spectrum of $x(t)$.

- $X(f)$ may not exist for most sample functions of a random process.
  - The Fourier transform exists for absolutely integrable functions.
  - The Fourier-Plancherel transform exists for square integrable functions.
  - A sample function may be neither absolutely integrable nor square integrable.
  - A sample function may be a power signal.
  - the power spectral density (PSD) or power density spectrum of a random process

- Energy Spectral Density (ESD) of a Finite-Energy Deterministic CT Signal
  - Definition. The energy of a deterministic CT signal $x(t)$ is defined as

$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t.$$

    * The energy is a non-negative real number.
  - Definition. If $0 < E_{xx} < \infty$, then $x(t)$ is called a finite-energy signal or an energy signal.
  - Definition. When $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(f)$, the energy spectral density (ESD) of an energy signal $x(t)$ is defined as

$$|X(f)|^2.$$

* The ESD is a non-negative real function of f. Moreover, it is symmetric if $x(t)$ is real.

* The integration of the ESD leads to E_{xx} because

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f.$$

* Fig. Bandpass filtering of $x(t)$ at f_0 with bandwidth ϵ

$$\begin{aligned}\int_{-\infty}^{\infty} |y(t)|^2 \, \mathrm{d}t &= \int_{-\infty}^{\infty} |Y(f)|^2 \, \mathrm{d}f \\ &= \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 \, \mathrm{d}f \\ &\cong \epsilon |X(f_0)|^2\end{aligned}$$

by Parseval's theorem.

* In general, the ESD does NOT fully characterize the signal. It partly characterizes the signal in the FD because $x(t)$ cannot be recovered from $|X(f)|^2$.

- Instantaneous Power vs. Average Power
  - Energy revisited:

$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t.$$

    * power = energy per unit time
    * energy = power times time
    * Thus, $|x(t)|^2$ is called the instantaneous power of $x(t)$.
    * The instantaneous power is a non-negative real number.

Def. The time average of a deterministic CT signal $x(t)$ is defined by

$$\overline{x} \triangleq \lim_{\Delta \to \infty} \frac{1}{2\Delta} \int_{-\Delta+C}^{\Delta+C} x(t)\, dt$$

$\langle x(t) \rangle$

$A[x(t)]$

Example 1. Given $x(t) = x(t+T)$, $\forall t$,

$$\bar{x} = \frac{1}{T}\int_{\langle T \rangle} x(t)\,dt$$

↑ It involves no limit operation

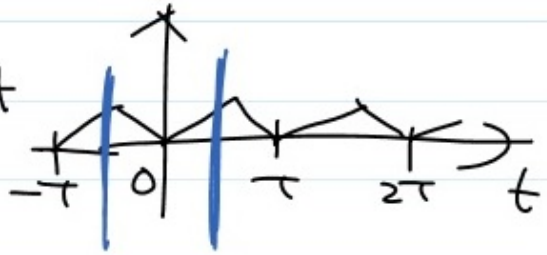
If $\bar{x}$ exists, then

$$\bar{x} = \lim_{\Delta \to \infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} x(t)\,dt$$

$$= \lim_{N \to \infty} \frac{1}{NT}\int_{-\frac{TN}{2}}^{\frac{NT}{2}} x(t)\,dt$$

$$= \lim_{N \to \infty} \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)\,dt$$

$$= \frac{1}{T}\int_{\langle T \rangle} x(t)\,dt$$



Example 2.

Given $x(t) = \begin{cases} x(t), & -\frac{I}{2} \le t < \frac{I}{2} \\ 0, & \text{elsewhere,} \end{cases}$

$$\bar{x} = \lim_{\Delta \to \infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} x(t)\,dt \quad \forall\ \Delta \ge \frac{I}{2}$$

$$= \lim_{\Delta \to \infty} \frac{1}{2\Delta} \underbrace{\int_{-\frac{I}{2}}^{\frac{I}{2}} x(t)\,dt}_{\text{If finite, then}}$$

$$= 0.$$

$\Rightarrow$ If $\int_{-\infty}^{\infty} x(t)\,dt < \infty$, then $\bar{x} = 0$

– Definition. The average power of a deterministic CT signal $x(t)$ is defined as

$$P_{xx} \triangleq \mathbb{A}\{|x(t)|^2\} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} |x(t)|^2 \mathrm{d}t.$$

where $\mathbb{A}\{\cdot\}$ denotes the time-average.

∗ The average power is a non-negative real number.

– Definition. If $0 < P_{xx} < \infty$, then $x(t)$ is called a finite-power signal or a power signal.

– Note that

$$0 < P_{xx} < \infty \implies E_{xx} = \infty$$
$$0 < E_{xx} < \infty \implies P_{xx} = 0$$

- Power Spectral Density (PSD) of a Finite-Power Deterministic CT Signal

– Definition. When

$$x_\Delta(t) = \begin{cases} x(t), & -\Delta < t < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

and $x_\Delta(t) \overset{\mathcal{F}}{\leftrightarrow} X_\Delta(f)$, the power spectral density (PSD) of a power signal $x(t)$ is defined as

$$S_{xx}(f) \triangleq \lim_{\Delta\to\infty} \frac{|X_\Delta(f)|^2}{2\Delta}.$$

∗ The PSD is a non-negative real function of f. If $x(t)$ is real, then it is symmetrical.

∗ Note that the PSD is the limit of the ESD $|X_\Delta(f)|^2$ divided by time 2Δ.

∗ The integration of the PSD leads to P_{xx} because

$$\begin{aligned} P_{xx} &= \lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{|x(t)|^2}{2\Delta} \mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{|x_\Delta(t)|^2}{2\Delta} \mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{|X_\Delta(f)|^2}{2\Delta} \mathrm{d}f \\ &= \int_{-\infty}^{\infty} \lim_{\Delta\to\infty} \frac{|X_\Delta(f)|^2}{2\Delta} \mathrm{d}f \\ &= \int_{-\infty}^{\infty} S_{xx}(f) \mathrm{d}f \quad \text{(if the limit exists)} \end{aligned}$$

∗ In general, the PSD does NOT fully characterize the signal. It partly characterizes the signal in the FD because $x(t)$ cannot be recovered from $S_{xx}(f)$.

- Cross-Power Spectral Density (xPSD) of Two Finite-Power Deterministic CT Signals
  - Q. Given two finite-power deterministic CT power signals $x(t)$ and $y(t)$, what is the PSD of $w(t) = x(t) + y(t)$?
  - A.
    - In general, $P_{ww} \neq P_{xx} + P_{yy}$.
      - Definition. The cross power between power signals $x(t)$ and $y(t)$ is defined as
$$P_{xy} \triangleq \mathbb{A}\{x(t)y^*(t)\}.$$
      - In general, the cross power is complex-valued. If $x(t)$ and $y(t)$ are real, then it is a real number that is positive, negative, or zero.

      - Thus, $P_{ww} = P_{xx} + P_{xy} + P_{yx} + P_{yy} = P_{xx} + 2\,\mathrm{Re}\{P_{xy}\} + P_{yy}$.

* Similarly, $S_{ww}(f) \neq S_{xx}(f) + S_{yy}(f)$ in general.
  - Definition. The cross-power spectral density (xPSD) between power signals $x(t)$ and $y(t)$ is defined as

$$S_{xy}(f) \triangleq \lim_{\Delta\to\infty} \frac{X_\Delta(f)Y_\Delta^*(f)}{2\Delta}.$$

  - In general, the xPSD is a complex-valued function. If $x(t)$ and $y(t)$ are real, then it is a conjugate symmetrical function.

* Thus, $S_{ww}(f) = S_{xx}(f) + S_{xy}(f) + S_{yx}(f) + S_{yy}(f) = S_{xx}(f) + 2\,\mathrm{Re}\{S_{xy}(f)\} + S_{yy}(f)$.

- Time Average of a CT Random Process
  - Q. Given a CT random process $X(t)$, what is the time average $\mathbb{A}\{X(t)\}$ of it?
  - A. It is a random variable because

$$\mathbb{A}\{X(t)\} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} X(t)\mathrm{d}t$$

    is a scalar with probabilistic description.

    * cf. The ensemble average or statistical average $\mathbb{E}\{X(t)\}$ is a deterministic CT signal.

    * Example. Given $\Pr(X(t)=\cos(2\pi t))=\Pr(X(t)=0)=\frac{1}{2}$, find $\mathbb{E}\{|X(t)|^2\}$ and $\mathbb{A}\{|X(t)|^2\}$.

$$\mathbb{E}\{|X(t)|^2\} = \frac{1}{2}\cos^2(2\pi t)$$

      and

$$\Pr(\mathbb{A}\{|X(t)|^2\}=1/2)=\Pr(\mathbb{A}\{|X(t)|^2\}=0)=1/2.$$

- PSD of a CT Random Process
  - Definition. The average power of a CT random process $X(t)$ is defined as

$$P_{XX} \triangleq \mathbb{A}[\mathbb{E}\{|X(t)|^2\}].$$

    * This definition is backward compatible with that for a deterministic CT signal $x(t)$.
    * It is the time average of the ensemble average of $|X(t)|^2$.
    * $P_{XX} = \mathbb{E}[\mathbb{A}\{|X(t)|^2\}]$.
    * Thus, it is a deterministic non-negative real number.

    * Example.

– Definition. When

$$X_\Delta(t) = \begin{cases} X(t), & -\Delta < t < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

and $X_\Delta(t) \overset{\mathcal{F}}{\leftrightarrow} X_\Delta(f)$, the PSD of a CT random process $X(t)$ is defined as

$$S_{XX}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\big[|X_\Delta(f)|^2\big]}{2\Delta}$$

* This definition is backward compatible with that for a deterministic CT signal $x(t)$.
* The PSD is a non-negative real function of f. If $X(t)$ is real, then it is symmetrical.
* Note that the PSD is the limit of $\mathbb{E}\{|X_\Delta(f)|^2\}$ divided by time 2Δ.

∗ The integration of the PSD leads to P_{XX} because

$$\begin{aligned}
P_{XX} &\triangleq \mathbb{A}\big[\mathbb{E}\big[|X(t)|^2\big]\big] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} \mathbb{E}\big[|X(t)|^2\big]\ \mathrm{d}t \\
&= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{\mathbb{E}\big[|X_\Delta(t)|^2\big]}{2\Delta}\,\mathrm{d}t \\
&= \lim_{\Delta\to\infty} \mathbb{E}\left[\int_{-\infty}^{\infty} \frac{|X_\Delta(f)|^2}{2\Delta}\,\mathrm{d}f\right] \\
&= \int_{-\infty}^{\infty} \lim_{\Delta\to\infty} \frac{\mathbb{E}\big[|X_\Delta(f)|^2\big]}{2\Delta}\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty} S_{XX}(f)\,\mathrm{d}f
\end{aligned}$$

∗ In general, the PSD does NOT fully characterize the CT random process. It partly characterizes the signal in the FD.

- xPSD of Two CT Random Processes
  - Q. Given two CT random processes $X(t)$ and $Y(t)$, what is the PSD of $W(t) = X(t) + Y(t)$?
  - A.
    - In general, $P_{WW} \neq P_{XX} + P_{YY}$.
      - Definition. The cross power between CT random processes $X(t)$ and $Y(t)$ is defined as
$$P_{XY} \triangleq \mathbb{A}[\mathbb{E}\{X(t)Y^*(t)\}].$$
      - This definition is backward compatible with that for deterministic CT signals $x(t)$ and $y(t)$.
      - In general, the cross power is complex-valued. If $X(t)$ and $Y(t)$ are real, then it is a real number that is positive, negative, or zero.
      - Thus, $P_{WW} = P_{XX} + P_{XY} + P_{YX} + P_{YY} = P_{XX} + 2\,\mathrm{Re}\{P_{XY}\} + P_{YY}$.
    - Similarly, $S_{WW}(f) \neq S_{XX}(f) + S_{YY}(f)$ in general.
      - Definition. The xPSD between two CT random processes $X(t)$ and $Y(t)$ is defined as
$$S_{XY}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\{X_\Delta(f)Y_\Delta^*(f)\}}{2\Delta}.$$
      - In general, the xPSD is a complex-valued function. If $X(t)$ and $Y(t)$ are real, then it is a conjugate symmetrical function.

* Thus, $S_{WW}(f) = S_{XX}(f) + S_{XY}(f) + S_{YX}(f) + S_{YY}(f) = S_{XX}(f) + 2\,\mathrm{Re}\{S_{XY}(f)\} + S_{YY}(f)$.

- Q. Why is the PSD so important in wireless communications, though it partially characterizes a random process?
- A.
  - The signal transmitted by a Tx is well modeled by a CT real-valued bandpass random process.
  - Regulatory organizations impose restrictions on the peak power, the average power, the PSD, etc. of the transmitted signal.
  - spectral mask
  - Thus, it is crucial to design standards and systems that do not violate such restrictions.
- Q. Recall that the PSD of a CT random process $X(t)$ is defined as

$$S_{XX}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\big[|X_\Delta(f)|^2\big]}{2\Delta}.$$

  Is there any alternative form with which we can easily compute the PSD?

  - Notice that the PSD is related only to the second-order moment of $X(t)$.

– The 2nd-order TD characterization is done by the auto-correlation function defined as

$$R_{XX}(t+\tau,t) \triangleq \mathbb{E}\{X(t+\tau)X^*(t)\}.$$

– The question is to find the relation between $R_{XX}(t+\tau,t)$ and $S_{XX}(f)$.

- A. The PSD can be found from the auto-correlation function as

$$S_{XX}(f) = \mathcal{F}\{\mathbb{A}[R_{XX}(t+\tau, t)]\}\,.$$

  - $S_{XX}(f) = \mathcal{F}\{\mathbb{A}[\mathbb{E}\{X(t+\tau)X^*(t)\}]\}$
  - The PSD is NOT affected by the complementary/pseudo auto-correlation function $\tilde{R}_{XX}(t+\tau, t)$.
  - $\mathbb{A}[R_{XX}(t+\tau, t)]$ is a function only of τ and often denoted by $\mathcal{R}_{XX}(\tau)$.
  - $\mathcal{R}_{XX}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XX}(f)$

- (Wiener-Khinchin Theorem) If $X(t)$ is WSS, then

$$R_{XX}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XX}(f),$$

i.e.,

$$\begin{cases} S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\ R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f) e^{\mathrm{j}2\pi f\tau}\,\mathrm{d}f. \end{cases}$$

– Example. Find the PSD of

$$X(t) = \cos(2\pi f_0 t) = \frac{e^{\mathrm{j}2\pi f_0 t} + e^{-\mathrm{j}2\pi f_0 t}}{2}, \forall t$$
$$\overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0).$$

∗ $X(t)$ is deterministic, so that no statistical average is required.

∗ Since $R_{XX}(t+\tau, t)$ is given by

$$\begin{aligned} R_{XX}(t+\tau, t) &= \cos 2\pi f_0(t+\tau) \cos 2\pi f_0(t) \\ &= \frac{1}{2}\left\{\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0 \tau)\right\}, \end{aligned}$$

$\mathcal{R}_{XX}(\tau)$ is obtained as

$$\mathbb{A}[R_{XX}(t+\tau, t)] = \frac{1}{2}\cos(2\pi f_0 \tau).$$

Thus,

$$S_{XX}(f) = \frac{1}{4}\delta(f - f_0) + \frac{1}{4}\delta(f + f_0).$$

– (Generalization) The PSD of $X(t) = \sum_{i \in I} a_i e^{\mathrm{j}2\pi f_i t}$ is given by

$$S_{XX}(f) = \sum_{i \in I} |a_i|^2 \delta(f - f_i).$$

∗ Fig. Graphs of $X(f)$ and $S_{XX}(f)$

∗ The PSD is NOT a full characterization of the random process. Later, we will see that it is the case when $X(t)$ is a real-valued WSS Gaussian random process.

• Q. Recall that the xPSD of two CT random processes $X(t)$ and $Y(t)$ is defined as

$$S_{XY}(f) \triangleq \lim_{\Delta \to \infty} \frac{\mathbb{E}\{X_\Delta(f) Y_\Delta^*(f)\}}{2\Delta}.$$

Is there any alternative form with which we can easily compute the xPSD?

- A. The xPSD can be found from the cross-correlation function as

$$S_{XY}(f) = \mathcal{F}\{\mathbb{A}[R_{XY}(t+\tau, t)]\}\,.$$

  - $S_{XY}(f) = \mathcal{F}\{\mathbb{A}[\mathbb{E}\{X(t+\tau)Y^*(t)\}]\}$
  - The PSD is NOT affected by the complementary/pseudo cross-correlation function $\tilde{R}_{XY}(t+\tau, t)$.
  - $\mathbb{A}[R_{XY}(t+\tau, t)]$ is a function only of τ and often denoted by $\mathcal{R}_{XY}(\tau)$.
  - $\mathcal{R}_{XY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XY}(f)$
  - If $X(t)$ and $Y(t)$ are jointly WSS, then

$$R_{XY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XY}(f),$$

- Proof.

$$
\begin{aligned}
S_{XX}(f) &= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\big[|X_\Delta(f)|^2\big] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\big[X_\Delta(f)\,(X_\Delta(f))^*\big] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[\left(\int_{-\infty}^{\infty} X_\Delta(t_1)e^{-\mathrm{j}2\pi f t_1}\,\mathrm{d}t_1\right)\left(\int_{-\infty}^{\infty} X_\Delta(t_2)e^{-\mathrm{j}2\pi f t_2}\,\mathrm{d}t_2\right)^*\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta} \mathbb{E}[X(t_1)X^*(t_2)]\,e^{-\mathrm{j}2\pi f(t_1-t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\int_{-\Delta-t}^{\Delta-t} \mathbb{E}[X(t+\tau)X^*(t)]\,e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta-t}^{\Delta-t}\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \lim_{\Delta\to\infty} \int_{-\Delta-t}^{\Delta-t} \frac{1}{2\Delta}\left(\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)\,\mathrm{d}t\right)e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty} \left(\lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)\,\mathrm{d}t\right)e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\
&= \mathcal{F}\{\mathbb{A}[R_{XX}(t+\tau,t)]\}
\end{aligned}
$$

- Properties of $S_{XX}(f)$ and $S_{XY}(f)$
  - Use either the definition or the theorem.
  - (real, non-negative PSD) $S_{XX}(f) \geq 0, \forall f$.
  - $\dot{X}(t) \triangleq \dfrac{\mathrm{d}}{\mathrm{d}t} X(t)$
    $S_{\dot{X}\dot{X}}(f) = (2\pi f)^2 S_{XX}(f)$
  - (conjugate pair) $S_{XY}(f) = S^*_{YX}(f), \forall f$.
    - $\mathrm{Re}\{S_{XY}(f)\} = \mathrm{Re}\{S_{YX}(f)\}$ and $\mathrm{Im}\{S_{XY}(f)\} = -\mathrm{Im}\{S_{YX}(f)\}$.
  - If $X(t)$ and $Y(t)$ are orthogonal, i.e., $R_{XY}(t+\tau, t) = 0, \forall t, \forall \tau$, then $S_{XY}(f) = 0, \forall f$. However, the converse is not true in general.

- If $X(t)$ and $Y(t)$ are jointly WSS, then
  - $R_{XX}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XX}(f)$
  - $R_{YY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{YY}(f)$
  - $R_{XY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XY}(f)$
  - $R_{YX}(\tau) = R^*_{XY}(-\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{YX}(f) = S^*_{XY}(f)$
  - $P_{XX} = R_{XX}(0), P_{YY} = R_{YY}(0), P_{XY} = R_{XY}(0) = R^*_{YX}(0) = P^*_{YX}$

- If $X(t)$ and $Y(t)$ are real-valued, then
  - (real, non-negative, symmetric PSD) $S_{XX}(f) = S_{XX}(-f), \forall f$

  - (conjugate symmetrical xPSD) $S_{XY}(f) = S^*_{XY}(-f), \forall f$

- Two most important CT random processes
  - $\begin{cases} \text{continuous random process: CT Gaussian random process} \\ \text{discrete random process: Poisson random process} \end{cases}$

  Fig.

– Definition. A real-valued CT random process $X(t)$ is Gaussian if $\forall N \in \mathbb{N}$ $\forall (t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$

the joint characteristic function of $\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_N) \end{bmatrix}$ is that of a real-valued Gaussian random vector.

– Lemma. $X(t)$ and $R_{XX}(t+\tau, t)$ fully characterized a real Gaussian random process.

– Theorem. If a real-valued Gaussian random process $X(t)$ is WSS, then it is strictly stationary.

(Proof)

$$\Phi_{X(t_1),X(t_2),\cdots,X(t_N)}(\underline{\omega}) = \mathbb{E}\left[e^{\mathrm{j}\underline{\omega}^T \underline{X}}\right]$$

$$= \exp\left(\mathrm{j}\underline{\omega}^T \underline{\mu} - \frac{\underline{\omega}^T C \underline{\omega}}{2}\right)$$

$$\underline{\mu} = \begin{bmatrix} \mathbb{E}[X(t_1)] \\ \mathbb{E}[X(t_2)] \\ \vdots \\ \mathbb{E}[X(t_N)] \end{bmatrix} = \mu \underline{1}$$

$$[C]_{i,j} = \mathbb{E}[(X(t_i) - \overline{X}(t_i))(X(t_j) - \overline{X}(t_j))]$$
$$= C_{XX}(t_i - t_j)$$

- Bandwidth of the PSD

  Fig. (real baseband r.p.)

  Fig. (real passband r.p.)

  RMS bandwidth

  1. $W_{\rm rms} \triangleq \sqrt{\int_{-\infty}^{\infty} f^2 \frac{S_{XX}(f)}{P_{XX}}\,{\rm d}f}$
  $$= \sqrt{\frac{\int_{-\infty}^{\infty} f^2 S_{XX}(f)\,{\rm d}f}{\int_{-\infty}^{\infty} S_{XX}(f)\,{\rm d}f}}$$
  2. $W_{\rm rms} \triangleq 2\sqrt{\int_{0}^{\infty} (f-\overline{f})^2 \frac{S_{XX}(f)}{P_{XX}/2}\,{\rm d}f}$

     where $\overline{f} = \int_{0}^{\infty} f \frac{S_{XX}(f)}{P_{XX}/2}\,{\rm d}f$

     $W_{\rm rms} = 2\sqrt{\frac{\int_0^{\infty}(f-\overline{f})^2 S_{XX}(f)\,{\rm d}f}{\int_0^{\infty} S_{XX}(f)\,{\rm d}f}}$

     where $\overline{f} = \int_{0}^{\infty} f S_{XX}(f)\,{\rm d}f$

Gaussian random process의 Peak-to-average power ratio가 W_{rms}의 함수!