HW09

Solve the following problems.

Problem 1. Suppose that a random variable X is defined on the probability space $(S, \mathcal{B}, P)$ with the sample space $S = \{s \in \mathbb{R} : 0 \leq s \leq 1\}$, the Borel σ-algebra $\mathcal{B}$ for the sample space S, and the Lebesgue measure P. Answer the following questions.

(a) Show that if $X(s) = 1$ then $\{s \in S : X(s) = 1\}) = S$.

(b) Show that if $X(s) = 0$ for $s = 0$ and $X(s) = 1$ for $s > 0$ then $X = 1$ with probability 1, i.e., $P(X = 1) \triangleq P(\{s \in S : X(s) = 1\}) = 1$.

(c) Show that if $X(s) = s$ then X is a uniform random variable on $[0, 1]$, i.e., $F_X(x) = x$ for $0 \leq x \leq 1$, $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 1$.

(d) Show that if $X(s) = 2(1 - s)$ then X is a uniform random variable on $[0, 2]$, i.e., $F_X(x) = x/2$ for $0 \leq x \leq 2$, $F_X(x) = 0$ for $x < 0$, and $F_X(x) = 1$ for $x > 2$.

Problem 2. Suppose that all the elements of the sequence of random variables $X_n(s)$ are defined on the probability space $(S, \mathcal{B}, P)$ with the sample space $S = \{s \in \mathbb{R} : 0 \leq s \leq 1\}$, the Borel σ-algebra $\mathcal{B}$ for the sample space S, and the Lebesgue measure P. When $X(s) = 1$ and $X_n(s) = 1 + u(s - 1 + 1/n)$, answer the following questions.

(a) Show that X_n does not converge to X everywhere.

(b) Show that X_n converges to X almost everywhere, i.e., $P(\{s \in S : X_n(s) \to X(s)\}) = 1$.

(c) Show that $\int_0^1 |X_n(s) - X(s)|^2 ds = 1/n \to 0$.

(d) Show that X_n converges to X in the p-th mean for any $p \geq 1$, i.e., $\int_0^1 |X_n(S) - X(s)|^p ds \to 0$ for any $p \geq 1$.

(e) Show that, for $\varepsilon = 1/10$, $P(\{s \in S : |X_n(s) - X(s)| > \varepsilon\}) = 1/n \to 0$.

(f) Show that X_n converges to X in measure, i.e., $\forall \varepsilon > 0, P(\{s \in S : |X_n(s) - X(s)| > \varepsilon\}) \to 0$.

(g) Show that X_n is a discrete random variable with $P(X_n = 1) = 1 - 1/n$ and $P(X_n = 2) = 1/n$.

(h) Show that the CDF $F_{X_n}(x)$ of X_n converges to the CDF $F_X(x)$ of X as $n \to \infty$.

Problem 3. Suppose that all the elements of the sequence of random variables $X_n(s)$ are defined on the probability space $(S, \mathcal{B}, P)$ with the sample space $S = \{s \in \mathbb{R} : 0 \leq s \leq 1\}$, the Borel σ-algebra $\mathcal{B}$ for the sample space S, and the Lebesgue measure P. When $X(s) = 1$ and $X_n(s) = 1 + nu(s - 1 + 1/n)$ for some $q > 0$, answer the following questions.

(a) Show that X_n does not converge to X everywhere.

(b) Show that X_n converges to X almost everywhere, i.e., $P(\{s \in S : X_n(s) \to X(s)\}) = 1$.

(c) Show that X_n does not converge to X in the p-th mean for any $p \geq 1$.

(d) Show that X_n converges to X in measure, i.e., $\forall \varepsilon > 0, P(\{s \in S : |X_n(s) - X(s)| > \varepsilon\}) \to 0$.

(e) Show that X_n is a discrete random variable with $P(X_n = 1) = 1 - 1/n$ and $P(X_n = 1 + n) = 1/n$.

(f) Show that the CDF $F_{X_n}(x)$ of X_n converges to the CDF $F_X(x)$ of X as $n \to \infty$.

Problem 4. Suppose that all the elements of the sequence of random variables $X_n(s)$ are defined on the probability space $(S, \mathcal{B}, P)$ with the sample space $S = \{s \in \mathbb{R} : 0 \leq s \leq 1\}$, the Borel σ-algebra $\mathcal{B}$ for the sample space S, and the Lebesgue measure P. When $X(s) = 1$ and $X_n(s) = 1 + \sqrt{n}u(s - 1 + 1/n)$ for some $q > 0$, answer the following questions.

(a) Show that X_n does not converge to X everywhere.

(b) Show that X_n converges to X almost everywhere, i.e., $P(\{s \in S : X_n(s) \to X(s)\}) = 1$.

(c) Show that X_n does not converge to X in the MS sense.

(d) Show that X_n converge to X in the p-th mean for $1 \leq p < 2$.

(e) Show that X_n does not converge to X in the p-th mean for $p > 2$.

(f) Show that X_n converges to X in measure, i.e., $\forall \varepsilon > 0, P(\{s \in S : |X_n(s) - X(s)| > \varepsilon\}) \to 0$.

(g) Show that X_n is a discrete random variable with $P(X_n = 1) = 1 - 1/n$ and $P(X_n = 1 + \sqrt{n}) = 1/n$.

(h) Show that the CDF $F_{X_n}(x)$ of X_n converges to the CDF $F_X(x)$ of X as $n \to \infty$.