HW13

A. From http://cisl.postech.ac.kr/lecture/574_2015sp/15exam03v150624a.pdf

8. (g)--(h)

B. From http://cisl.postech.ac.kr/lecture/574_2015sp/15quiz09v150521a.pdf

3.

C. Solve the following problems.

Problem 2. (15 points) Suppose that $X(t)$ is a complex-valued white noise with autocorrelation function

$$R_{XX}(t_1, t_2) \triangleq \mathbf{E}[X(t_1)X^*(t_2)] = q(t_1)\delta(t_1 - t_2).$$

When $X(t)$ is filtered by a linear time-invariant system with impulse response $h(t)$, i.e.,

$$Y(t) = h(t) * X(t) \triangleq \int_{-\infty}^{\infty} h(\tau)X(t-\tau)d\tau$$

answer the following questions.

(a) (5 points) Find $R_{XY}(t_1, t_2)$ in terms of $q(t)$ and $h(t)$ without leaving the convolution operator $*$ in your answer.

(b) (5 points) Using the result in (a), find $R_{YY}(t_1, t_2)$ in terms of $q(t)$ and $h(t)$ without leaving the convolution operator $*$ in your answer.

(c) (3 points) Using the result in (b), find $\mathbf{E}[|Y(t)|^2]$ in terms of $q(t)$ and $h(t)$ without leaving the convolution operator $*$ in your answer.

(d) (2 points) Using the result in (c), show that

$$\mathbf{E}[|Y(t)|^2] = q(t) * |h(t)|^2.$$

Problem 3. (15 points) Suppose that $X(t)$ is a Poisson random process with arrival rate λ and $Y(t)$ is given by

$$Y(t) = \frac{dX(t)}{dt}.$$

Answer the following questions.

(a) (3 points) When $X(t)$ is rewritten as

$$X(t) = \sum_i u(t - t_i)$$

where t_i is the ith arrival time and $u(t)$ is the unit-step function, find $Y(t)$ in terms of t_i and the Dirac delta function $\delta(t)$.

(b) (2 points) Using the result in (a), sketch a sample path of $X(t)$ and the corresponding sample path of $Y(t)$.

(c) (5 points) State the 1st-order probability mass function of $X(t)$ and derive the mean function $\mathbf{E}[X(t)]$.

(d) (5 points) Using the result in (c), find and sketch the mean path of $Y(t)$.

Problem 4. (20 points) Suppose that $X(t)$ is a wide-sense stationary (WSS) random process with mean μ_X and autocorrelation $R_{XX}(\tau)$. When $Y(t)$ is defined as

$$Y(t) = \frac{1}{2T} \int_{t-T}^{t+T} X(\tau) d\tau$$

where $T > 0$ is a constant, answer the following questions.

(a) (5 points) Find the mean and variance of $Y(t)$ in terms of T, μ_X, and $R_{XX}(\tau)$.

(b) (5 points) Find the autocorrelation and autocovariance functions of $Y(t)$ in terms of T, μ_X, and $R_{XX}(\tau)$.

(c) (5 points) Prove or disprove that $Y(t)$ is WSS.

(d) (5 points) When the autocovariance function of $X(t)$ is absolutely integrable, how can you approximate $Y(t)$ when T is sufficiently large? Justify your answer.

Problem 5. (30 points) Suppose that $Y(t)$ is a complex-valued random process given by

$$Y(t) = \sum_{l=1}^{L} \alpha_l X_i e^{\mathrm{j}2\pi f_l (t-\tau_l)}$$

where X_i's are i.i.d. zero-mean proper-complex Gaussian random variables with unit variance and α_l's and τ_l's are constants.

When f_l's are distinct positive constants, answer the following questions.

(a) (5 points) Find the mean, variance, and pseudo-variance of $Y(t)$.

(b) (5 points) Find the autocorrelation $R_{YY}(t+\tau, t)$ and complementary autocorrelation $\widetilde{R}_{YY}(t+\tau, t)$ functions of $Y(t)$.

(c) (5 points) Prove or disprove that $Y(t)$ is a proper-complex Gaussian random process.

(d) (5 points) Prove or disprove that $Y(t)$ is WSS.