HW14

Solve the following problems.

Problem 5. (30 points) Suppose that $Y(t)$ is a complex-valued random process given by

$$Y(t) = \sum_{l=1}^{L} \alpha_l X_i e^{\mathrm{j}2\pi f_l (t-\tau_l)}$$

where X_i's are i.i.d. zero-mean proper-complex Gaussian random variables with unit variance and α_l's and τ_l's are constants.

When f_l's are distinct positive constants, answer the following questions.

(e) (5 points) Find the time average $\mathcal{R}_{YY}(\tau)$ of $R_{YY}(t+\tau, t)$ averaged over t.

(f) (5 points) Using

$$S_{YY}(f) = \int_{-\infty}^{\infty} \mathcal{R}_{YY}(\tau) e^{-\mathrm{j}2\pi f \tau} d\tau,$$

find the PSD of $Y(t)$.

Problem 5. (20 points) Suppose that Z and W are jointly distributed random variables. Answer the following questions.

(a) (5 points) Prove the Schwarz inequality

$$\{E[ZW]\}^2 \leq E[Z^2]E[W^2]$$

and also find the condition for the equality to hold.

(b) (5 points) Using the result in (a), when X and Y are jointly distributed random variables with non-zero variances, show that

$$-1 \leq \rho_{XY} \leq 1,$$

and also find the conditions for the equalities to hold.

(c) (5 points) Using the result in (a), show that a continuous-time random process $X(t)$ is mean-square periodic with period $T > 0$, i.e.,

$$E[|X(t+T) - X(t)|^2] = 0, \forall t$$

if and only if

$$R_{XX}(t_1, t_2) = R_{XX}(t_1 + mT, t_2 + nT), \forall t, \forall m, \forall n.$$

(d) (5 points) Using the result in (a), show that a continuous-time wide-sense stationary random process $X(t)$ is mean-square continuous, i.e.,

$$E[|X(t+\varepsilon) - X(t)|^2] \to 0 \text{ as } \varepsilon \to 0,$$

$\forall t$, if and only if $R_{XX}(\tau)$ is continuous $\forall \tau$.

3. (25 points) Suppose that $X(t)$ is a Wiener process with $X(t) \sim \mathcal{N}(0, \alpha t)$ and $N(t)$ is another independent Wiener process with $N(t) \sim \mathcal{N}(0, \beta t)$ with $\alpha > \beta > 0$. When the random process $Y(t)$ is given by

$$Y(t) = X(t) + N(t),$$

answer the following questions.

(a) (5 points) Find the 1st-order PDF of $Y(t)$. Also find the mean path of $Y(t)$.

(b) (10 points) Find the 2nd-order PDF of $Y(t)$. Also find the auto-correlation function of $Y(t)$.

(c) (5 points) Find the cross-correlation function between $X(t)$ and $Y(t)$.

(d) (5 points) Prove or disprove that $X(t)$ and $Y(t)$ are jointly wide-sense stationary.