AWGN, ACGN, Band-Limited AWGN, Proper-Complex Band-Limited AWGN, Proper-Complex AWGN

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- Review. The received signal contains a noise $N(t)$ often modeled by **a real-valued bandpass wide-sense stationary (WSS) random process** called an additive white Gaussian noise (AWGN).
  - TD characterization: mean, auto- and cross-correlation functions
  - FD characterization: PSDs and xPSD

- Preview.
  - Rx front end and AWGN
  - real-valued additive colored Gaussian noise (ACGN)
  - real-valued band-limited AWGN
  - proper-complex band-limited AWGN
  - proper-complex AWGN

- Receiver Front End and AWGN
  - Consider a receiver front end with a single Rx antenna.
  - Fig.

    * At point O, there are a desired signal component $\tilde{X}(t)$ and an interference component $I(t)$.
      · The antenna is also a source of thermal noise. However, it is relatively small compared with that added by the LNA.
      · $\tilde{X}(t)$ and $I(t)$ are modeled by **real-valued CT bandpass random processes**.
    * At point A, after $\tilde{X}(t)$ and $I(t)$ being amplified, a **noise** $N(t)$ generated by the **LNA** is added.
    * At point B, the noise $N(t)$ is bandpass filtered to become $\tilde{N}(t)$.

– $N(t)$ is well modeled by **a real-valued WSS Gaussian random process** with PSD given by

$$S_{NN}(f) = \frac{2Rh|f|}{e^{\frac{h|f|}{kT}} - 1}, \begin{cases} S_{NN}(f) \to 0 & \text{as } f \to \infty \\ S_{NN}(f) \simeq 2RkT & \text{when } f \ll \infty \end{cases}$$

where $h = 6.62 \times 10^{-34}$ is Planck's constant,

$k = 1.3799 \times 10^{-23}$ is Boltzman's constant,

R is the resistance,

T is the temperature in Kelvin.

* Fig.

* Note that $e^x \simeq 1 + x$ for small x. Thus, the PSD is flat up to hundreds of GHz.
* Note that, since Gaussianity is assumed, the wide-sense stationarity implies strict-sense stationarity.

– $\tilde{N}(t)$ becomes **a real-valued WSS Gaussian random process** with PSD given by

$$S_{\tilde{N}\tilde{N}}(f) = |H(f)|^2 S_{NN}(f), \forall f.$$

* Usually, the BPF has an upper cut-off frequency much lower than hundreds of GHz.
  · In this case, $S_{\tilde{N}\tilde{N}}(f) \approx |H(f)|^2 2RkT, \forall f$.
  · As far as $N(t)$ is to be bandpass filtered, its PSD $S_{NN}(f)$ can be assumed flat for all f without loss of generality.
  · This **zero-mean WSS Gaussian random process with flat PSD** is called **additive white Gaussian noise (AWGN)**.
  · By convention, $S_{NN}(f) = N_0/2, \forall f$, where N_0 is called the one-sided PSD while $N_0/2$ is called the two-sided PSD.
* $\tilde{N}(t)$ is called a real-valued **additive colored Gaussian noise (ACGN)**.

- Def. A real-valued WSS Gaussian random process $N(t)$ with PSD given by

$$S_{NN}(f) = \frac{N_0}{2}, \forall f$$

is often called **a real-valued AWGN**.

– Fig.

– The auto-correlation function is

$$R_{NN}(\tau) = \qquad .$$

* Thus, it has power $P_{NN} =$.
* It does not exist in the real world. It is a mathematical model for engineering convenience.

- Q. When a real-valued AWGN $N(t)$ with two-sided PSD $N_0/2$ is filtered by an LTI sytem with impulse response $h(t)$, discuss the property of the output $\tilde{N}(t) = h(t) * N(t)$.
- A. The output $\tilde{N}(t)$ is
  - a real-valued WSS Gaussian random process with PSD

$$S_{\tilde{N}\tilde{N}}(f) = \frac{N_0}{2}|H(f)|^2,$$

  - and has the in-phase and the quadrature components $\tilde{N}_c(t)$ and $\tilde{N}_s(t)$
    - such that $\tilde{N}(t) = \tilde{N}_c(t)\cos(2\pi f_c t) - \tilde{N}_s(t)\sin(2\pi f_c t)$
    - that are identically distributed
    - baseband zero-mean WSS Gaussian random processes because ...
    - but not necessarily independent.
      - They are independent iff the PSD for positive frequency part is symmetrical with respect to f_c.

- Def. A real-valued WSS Gaussian random process $N(t)$ with PSD given by

$$S_{NN}(f) = \frac{N_0}{2}, \text{ for } |f - f_c| \leq \frac{W}{2},$$

for some $f_c \geq W/2$ is often called **a real-valued band-limited AWGN**.

  - Fig.

  - Q. Bandwidth?
  - Q. What if $f_c = W/2$?
  - The in-phase and the quadrature components $N_c(t)$ and $N_s(t)$ of $N(t) = N_c(t)\cos(2\pi f_c t) - N_s(t)\sin(2\pi f_c t)$ are
    * i.i.d.
    * real-valued
    * band-limited AWGN
    * with two-sided PSD N_0 for $|f| \leq W/2$.

- Def. A proper-complex WSS Gaussian random process $Z(t)$ with PSD given by

$$S_{ZZ}(f) = 2N_0, \text{ for } |f| \leq \frac{W}{2}$$

is often called **a proper-complex band-limited AWGN**.

  - It is the complex envelope of a real-valued band-limited AWGN with two-sided PSD $N_0/2$.
  - Fig.

  - Q. Bandwidth in complex baseband?
  - Q. Bandwidth in real passband?
  - The real and the imaginary parts $Z_c(t)$ and $Z_s(t)$ of $Z(t) = Z_c(t) + \mathrm{j}Z_s(t)$ are
    * i.i.d.
    * real-valued
    * band-limited AWGN
    * with two-sided PSD N_0, for $|f| \leq W/2$.

- Def. A proper-complex WSS Gaussian random process $N(t)$ with PSD given by

$$S_{NN}(f) = 2N_0, \forall f$$

is often called **a proper-complex AWGN**.

  - Fig.
  - The real and the imaginary parts are
    - i.i.d.
    - real-valued AWGN with two-sided PSD $N_0, \forall f$.
  - The auto-correlation function is

$$R_{NN}(\tau) = \qquad .$$

    - Thus, it has power $P_{NN} = \qquad .$
    - It does not exist in the real world. It is a mathematical model for engineering convenience.
    - It is okay, as far as $\tilde{N}_l(t)$ is to be filtered by a band-limited baseband LTI filter.
  - It has zero complementary auto-correlation function.

- Discussions:
  - real-valued band-limited AWGN approximation to colored Gaussian noise
    * $N(t)$ at point A and $\hat{N}(t)$ at point B are both additive colored Gaussian noise.

    * But, they are often modeled by an AWGN and a band-limited AWGN, respectively.
      · In some applications, the AWGN is one of the most important channel impairment in communications.
      · This **thermal** noise is unavoidable as far as EM wave in radio frequency band is used for wireless communications and it is received by an antenna and amplified by an LNA at the Rx.
      · If the signal strength is large enough, then no thermal noise is observed. However, even in this case, other channel impairments are often aggregated and modeled by an AWGN.

- proper-complex band-limited AWGN approximation to colored Gaussian noise
  - In complex baseband representation of the real bandpass received signal, the noise is still often modeled by a proper-complex AWGN, which is precisely speaking incorrect.
  - However, as far as it is to be immediately lowpass filtered, proper-complex AWGN modeling loses no generality.
  - In this case, the complex signal is called the **complex baseband equivalent** to the real bandpass received signal, not the complex envelope.

- Summary
  - real-valued AWGN
  - real-valued ACGN
  - real-valued band-limited AWGN
  - proper-complex band-limited AWGN
  - proper-complex AWGN