AWGN, ACGN, Band-Limited AWGN, Proper-Complex Band-Limited AWGN, Proper-Complex AWGN: Exercises

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- Review. The received signal contains a noise $N(t)$ often modeled by **a real-valued bandpass wide-sense stationary (WSS) random process** called an additive white Gaussian noise (AWGN).
  - Rx front end and AWGN
  - real-valued additive colored Gaussian noise (ACGN)
  - real-valued band-limited AWGN
  - proper-complex band-limited AWGN
  - proper-complex AWGN

- Preview. Exercises
  - Independence of in-phase and quadrature noise
  - Delay and phase sensitivity
  - Complex baseband representation and scaling

- Q1. When $X(t)$ is a real-valued WSS Gaussian random process with PSD $S_{XX}(f)$ given by

and an I-Q demodulator processes it as

answer the following questions.

(a) When $f_0 = f_c$, find $S_{X_cX_c}(f), S_{X_sX_s}(f)$, and $S_{X_sX_c}(f)$.

(b) When $f_0 = f_c$, are $X_c(t)$ and $X_s(t)$ independent processes?

(c) When $f_0 = f_c + \Delta/2$, find $S_{X_cX_c}(f), S_{X_sX_s}(f)$, and $S_{X_sX_c}(f)$.

(d) When $f_0 = f_c + \Delta/2$, are $X_c(t)$ and $X_s(t)$ independent processes?

- Q2. Given is a real-valued bandpass WSS Gaussian random process $X(t)$ such that

$$X(t) = \mathrm{Re}\{X_l(t)e^{\mathrm{j}2\pi f_c t}\}$$

with complex envelope $X_l(t) = X_c(t) + \mathrm{j}X_s(t)$, define $Y(t) = \mathrm{Re}\{X_l(t) e^{\mathrm{j}(2\pi f_c t + \theta)}\}$ with complex envelope $Y_l(t) = Y_c(t) + \mathrm{j}Y_s(t)$ and $Z(t) = \mathrm{Re}\{X_l(t-\Delta)e^{\mathrm{j}2\pi f_c(t-\Delta)}\}$ with complex envelope $Z_l(t) = Z_c(t) + \mathrm{j}Z_s(t)$, and answer the following questions.

(a) Show that the PSDs of $X(t), Y(t)$, and $Z(t)$ are invariant to the change of θ and Δ.

(b) Justify that $X_c(t), Y_c(t)$, and $Z_c(t)$ are identically distributed.

(c) Justify that $X_s(t), Y_s(t)$, and $Z_s(t)$ are identically distributed.

(d) Justify that $X_l(t), Y_l(t)$, and $Z_l(t)$ are identically distributed.

- Q3. Suppose that the received real-valued bandpass signal is given by
$$Y(t) = \sqrt{2P}X(t)\cos(2\pi f_c t + \theta) + N(t),$$
where $X(t)$ is a unit-power real-valued baseband random process with bandwidth W, $f_c \gg W$ and θ are known constants, and $N(t)$ is a real value band-limited AWGN with two-sided PSD $N_0/2$. Answer the following questions.

(a) Show that $\sqrt{2P}X(t)\cos(2\pi f_c t + \theta)$ has unit power.

(b) When the complex envelope of $Y(t)$ is modeled as
$$Y_1(t) = \sqrt{2P}X(t)e^{j\theta} + N_1(t),$$
describe $N_1(t)$.

(c) When the complex baseband equivalent to $Y(t)$ is modeled as
$$Y_2(t) = \sqrt{P}X(t)e^{j\theta} + N_2(t),$$
describe $N_2(t)$.

(d) When $Y(t)$ is coherently down-converted to a real baseband signal
$$Y_3(t) = \sqrt{P}X(t) + N_3(t),$$
describe $N_3(t)$.

- Q. Suppose that the received signal $Y(t)$ is modeled by a real-valued bandpass random process given by

$$Y(t) = \sqrt{2P}bs(t)\cos(2\pi f_c t + \theta) + N(t),$$

where b is an unknown data with $\Pr(b = 1) = \Pr(b = -1) = 1/2$, $s(t)$ is a real-valued baseband signal with energy T and bandwidth W, $f_c \gg W$ and θ are known constants, and $N(t)$ is a real value band-limited AWGN with two-sided PSD $N_0/2$. Answer the following questions.

(a) Find the complex baseband equivalent to the received signal $Y(t)$.

– ANS.

$$Y_l(t) = \sqrt{2P}bs(t)e^{\mathrm{j}\theta} + N_l(t),$$

where $N_l(t)$ is the proper-complex AWGN with PSD $2N_0$.

(b) Find the PDF of $N \triangleq N_l(t) * s(-t)|_{t=0}$.

– ANS. $N \sim \mathcal{CN}(0, 2N_0T)$.

(c) Find $\{Y_l(t) - N_l(t)\} * s(-t)|_{t=0}$.

– ANS. $\sqrt{2P}bTe^{\mathrm{j}\theta}$.

(d) Find the probability of event that $\mathrm{Re}\{Y_l(t) * s(-t)|_{t=0} \cdot e^{-\mathrm{j}\theta}\} \leq 0$ conditioned on $\{b = 1\}$.

– ANS.

$$P_b = Q\left(\sqrt{\frac{2PT}{N_0}}\right).$$