Digitally Modulated Signals and Their PSDs: ASK and PSK

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- Review. So far, the transmitted signal is just assumed a **real-valued bandpass** signal $x(t)$ or random process $X(t)$.
  - complex baseband representation of real-valued bandpass signal, random process, and LTI system
  - homodyne transmitter, heterodyne transmitter, quadrature modulator, polar modulator
  - TD and FD characterization of random processes: correlation functions, PSD, and xPSD
  - (PSD of a real-valued bandpass WSS random process)

- Preview.
  - Amplitude Shift Keying (ASK)
  - Phase Shift Keying (PSK)
  - PSDs

- Amplitude and Phase
  - Real-valued bandpass random process $X(t) = \mathrm{Re}\{X_l(t)e^{\mathrm{j}2\pi f_c t}\}$
    * Cartesian Coordinate $X_l(t) = X_c(t) + \mathrm{j}X_s(t)$

$$X(t) = X_c(t)\cos(2\pi f_c t) - X_s(t)\sin(2\pi f_c t)$$

where

· $X_c(t)$ is the amplitude of the in-phase carrier and

· $X_s(t)$ is the amplitude of the quadrature-phase carrier.

    * Polar Coordinate $X_l(t) = A(t)e^{\mathrm{j}\theta(t)}$

$$X(t) = A(t)\cos(2\pi f_c t + \theta(t))$$

where

· $A(t)$ is the amplitude of the carrier and

· $\theta(t)$ is the angle/phase of the carrier.

- Amplitude Shift Keying (ASK)
  - ASK is a special case of amplitude modulation where a digital message is embedded by **shifting (=abruptly changing) the amplitude of the carrier**.
  - $\theta(t) = \theta$ is constant but $A(t)$ is given by

$$A(t) = \sum_{n=-\infty}^{\infty} a[n] p_T(t - nT)$$

  where
    * $p_T(t)$ is the rectangular pulse with support $[0, T)$ and height 1,
    * $a[n] \in \mathcal{A}$ is a real-valued scalar, and
    * a finite set $\mathcal{A}$ of real numbers is called the **symbol constellation**.

– Remarks

* $1/T$ is the rate of change in amplitude.
  · T is often called the **symbol period**, and $1/T$ is often called the **symbol rate** in [symbols/sec].
  · Sometimes, baseband signaling with $f_c = 0$ is considered for ASK.
  · In bandpass signaling, $f_c \gg 1/T$ is chosen for ASK.

* $\mathcal{A} = \{0, 1\}$ or equivalent is called the on-off keying (OOK).
  · **In a narrow sense, the ASK means the OOK.**
  · **In a narrow sense, the PAM means the ASK described here.**
* $\mathcal{A} = \{-1, 1\}$ or equivalent is often called the 2-ASK, B-ASK, or the binary antipodal signaling.
  · $\mathcal{A} = \{-3, -1, 1, 3\}$ or equivalent is called the 4-ASK.
  · M-ASK can be similarly defined.
  · **constellation diagram**
  · **gray coding** vs. **natural binary coding** for $M = 2^m$

- Phase Shift Keying (PSK)
  - PSK is a special case of angle modulation where a digital message is embedded by **shifting (=abruptly changing) the angle of the carrier**.
  - $A(t) = A$ is constant but $\theta(t)$ is given by

$$\theta(t) = \sum_{n=-\infty}^{\infty} \theta[n] p_T(t - nT) + \theta$$

    where
    * $p_T(t)$ is again the rectangular pulse with support $[0, T)$ and height 1,
    * $\theta[n] \in \Theta \subset [0, 2\pi)$ is a real-valued scalar, and
    * $\Theta \subset [0, 2\pi)$ is a finite set of real numbers.

– Remarks

  * $1/T$ is the rate of change in phase.
    · T is often called the symbol period, and $1/T$ is often called the symbol rate in [symbols/sec].
    · $f_c = 0$ is not allowed for PSK in general.
    · In bandpass signaling, $f_c \gg 1/T$ is chosen for PSK.

  * If $\Theta = \{0, \pi\}$ or equivalent, then the PSK is called the binary phase shift keying (BPSK).
    · If $f_c = 0$, then there is no phase to shift but it is still often called BPSK.
  * $\Theta = \{\pi/4, 3\pi/4, 5\pi/4, 7\pi/4\}$ or equivalent is often called the **quaternary phase-shift keying (QPSK)**.
  * $\Theta = \{1 \cdot 2\pi/M, 2 \cdot 2\pi/M, ..., M \cdot 2\pi/M\}$: **M-ary phase-shift keying (MPSK)**

- A Unified View
  - Both ASK and PSK can be viewed as

$$X(t) = \mathrm{Re}\{X_l(t)e^{\mathrm{j}(2\pi f_c t+\theta)}\}$$

    where the complex envelope is given by

$$X_l(t) = \sum_{n=-\infty}^{\infty} d[n]p_T(t-nT).$$

    * In ASK, $(d[n]) = (a[n])$ is a sequence of symbols.
    * In PSK, $(d[n]) = (e^{\mathrm{j}\theta[n]})$ is a sequence of symbols.
    * $\mathcal{A} \ni d[n]$ is a symbol constellation.
    * $p_T(t)$ is again the rectangular pulse with support $[0, T)$ and height 1.
      - $1/T$ is the symbol transmission rate.
    * constellation diagram for PSK
    * gray coding vs. natural binary coding for $M(=2^m)$-PSK.

- Now, the only difference is the symbol constellation.
    - In ASK, $\mathcal{A}$ or equivalent is a finite set of distinct real numbers.
    - In PSK, $\mathcal{A}$ or equivalent is a finite set of distinct complex numbers on **the unit circle**.
        - 2-ASK with $\mathcal{A} = \{-1, 1\}$ is equivalent to BPSK, whether $f_c = 0$ or not.

- Q. Find the PSDs of ASK and PSK signals under the assumption that $(d[n])$ is a sequence of i.i.d. symbols with equally-likely **constellation points**.
  - Recall that the PSD of a real-valued random process $X(t)$ can be computed as

$$S_{XX}(f) = \mathcal{F}[\mathcal{A}\{R_{XX}(t+\tau, t)\}]$$

where

$$R_{XX}(t+\tau, t) = \mathbb{E}\{X(t+\tau)X(t)\}.$$

- Lemma. (General Result) If $f_c \gg$ twice the bandwidth of $X_l(t)$, then

$$S_{XX}(f) = \frac{S_{X_lX_l}(f - f_c) + S_{X_lX_l}(-(f + f_c))}{4}. \tag{1}$$

  – Proof. Note that

$$\begin{aligned}
\mathbb{E}\{X(t+\tau)X(t)\} &= \mathbb{E}\left[\mathrm{Re}\{X_l(t+\tau)e^{\mathrm{j}(2\pi f_c(t+\tau)+\theta)}\} \cdot \mathrm{Re}\{X_l(t)e^{\mathrm{j}(2\pi f_c t+\theta)}\}\right] \\
&= \ldots \\
&= \frac{\mathrm{Re}\left\{\tilde{R}_{X_lX_l}(t+\tau,t)e^{\mathrm{j}(2\pi f_c(2t+\tau)+\theta)}\right\}}{2} \\
&\quad + \frac{\mathrm{Re}\left\{R_{X_lX_l}(t+\tau,t)e^{\mathrm{j}2\pi f_c\tau}\right\}}{2}.
\end{aligned}$$

  – If $f_c \gg$ the bandwidth of the sample paths of $\tilde{R}_{X_lX_l}(t+\tau,t) = \mathbb{E}\{X_l(t+\tau)X_l^*(t)\}$, i.e., twice the bandwidth of $X_l(t)$, then the time average of the first term with respect to t is zero, regardless of the propriety of $X_l(t)$.

  – Thus, the time average becomes

$$\mathcal{A}\{R_{XX}(t+\tau,t)\} = \frac{\mathrm{Re}\left\{\mathcal{R}_{X_lX_l}(\tau)e^{\mathrm{j}2\pi f_c\tau}\right\}}{2},$$

  which immediately leads to Eq. (1).

- A. As assumed, $(d[n])$ is a sequence of i.i.d. symbols with equally-likely constellation points.
  - Then, $\mathbb{E}\{d[n]d^*[m]\} = \delta[n-m]$ if we set $\mathbb{E}\{|d[n]|^2\} = 1$ for convenience.
  - Thus, $R_{X_lX_l}(t+\tau, t) = ... = \sum_{n=-\infty}^{\infty} p_T(t+\tau-nT)p_T(t-nT)$, which is periodic in t with period T.
  - This leads to

$$\begin{aligned}
\mathcal{R}_{X_lX_l}(\tau) &= \frac{1}{T}\int_0^T \sum_{n=-\infty}^{\infty} p_T(t+\tau-nT)p_T(t-nT)dt \\
&= \frac{1}{T}\int_{-\infty}^{\infty} p_T(t+\tau)p_T(t)dt \\
&= ... \\
&= \frac{1}{T}p_T(\tau) * p_T(-\tau).
\end{aligned}$$

  - Therefore,

$$S_{X_lX_l}(f) = \frac{1}{T}\mathrm{sinc}^2(fT).$$

- PSD of ASK and PSK
  - Fig.
  - RMS bandwidth, null-to-null bandwidth, 3 dB bandwidth, η%-power bandwidth
  - BPSK vs. QPSK
    - They have the same PSD.
    - QPSK transmits twice many bits per sec.
    - As DSB wastes half the bandwidth compared to SSB, BPSK wastes half the bandwidth compared to QPSK.
  - PSD is proportional to sinc square, which decays at the rate of -20 dB/decade.
    - Can we have a better PSD?
    - pulse-shaped ASK = PAM
    - offset QPSK (OQPSK), minimum shift keying (MSK)
    - QAM