# *Digitally Modulated Signals and Their PSDs: PAM and QAM*

Prof. Joon Ho Cho
Department of Electrical Engineering, POSTECH

- Review. A digitally modulated transmitted signal $X(t)$ is assumed a **real-valued bandpass** random process.
  - PSD of complex envelope $X_l(t)$ and PSD of $X(t)$
  - ASK, OOK, M-ASK
  - PSK, BPSK, M-PSK
  - PSD of ASK and PSK with i.i.d. symbols and uniform constellation points

- Preview.
  - Pulse Amplitude Modulation (PAM)
  - Quadrature Amplitude Modulation (QAM)
  - PSDs of PAM and QAM signals with wide-sense stationary (WSS) symbol sequences

- Digitally modulated real-valued bandpass transmitted signal $X(t)$
  - TD
    - In many cases, it is not a WSS random process in general.
      - Its complex envelope is not WSS in general.
      - Its complex envelope is not proper in general.
    - In many cases, its complex envelope is a proper-complex wide-sense cyclostationary (WSCS) random process.
      - Def. A random process $Z(t)$ is WSCS if there exists $0 < T < \infty$ such that
        (i) $\mu_Z(t) \triangleq \mathbb{E}\{Z(t)\} = \mu_Z(t+T), \forall t$ and
        (ii) $R_{ZZ}(t,s) \triangleq \mathbb{E}\{Z(t)Z^*(s)\} = R_{ZZ}(t+T, s+T), \forall t, \forall s.$
  - FD

$$S_{XX}(f) = \frac{S_{X_l X_l}(f - f_c) + S_{X_l X_l}(-(f + f_c))}{4}$$

  - Example: ASK and PSK

* TD
  - When the data symbols are i.i.d. with mean zero and variance one, the complex envelope $X_l(t)$ is not WSS but WSCS because

$$R_{X_lX_l}(t,s) = \sum_{n=-\infty}^{\infty} p_T(t-nT)p_T(s-nT) = R_{X_lX_l}(t+T, s+T).$$

  - $X_l(t)$ is not proper for BPSK because $R_{X_lX_l}(t,s) = \tilde{R}_{X_lX_l}(t,s)$.
  - $X(t)$ is not wide-sense stationarity because

$$\mathbb{E}\{X(t+\tau)X(t)\} = \frac{\mathrm{Re}\left\{\tilde{R}_{X_lX_l}(t+\tau,t)e^{\mathrm{j}(2\pi f_c(2t+\tau)+\theta)}\right\}}{2} + \frac{\mathrm{Re}\left\{R_{X_lX_l}(t+\tau,t)e^{\mathrm{j}2\pi f_c\tau}\right\}}{2}.$$

* FD
  - When the data symbols are i.i.d. with mean zero and variance one, the PSD is given by

$$S_{X_lX_l}(f) = T\mathrm{sinc}^2(fT).$$

  - strict-sense bandwidth, null-to-null bandwidth, decay rate of sidelobes

- Pulse Amplitude Modulation (PAM)
  - Def. PAM is a pulse-shaped ASK, where the rectangular pulse is replaced by another waveform.
    * Thus, the transmitted signal is given by

$$X(t) = \mathrm{Re}\left\{ \left( \sum_{n=-\infty}^{\infty} a[n] p(t - nT) \right) e^{\mathrm{j}(2\pi f_c t + \theta)} \right\}$$

    * where
      · $p(t)$ is called the **transmit waveform** or the **transmit pulse**,
      · $T$ is often called the symbol period,
      · $1/T$ is called the symbol transmission rate or the baud rate, and
      · $\mathcal{A} \ni a[n]$ is a one-dimensional constellation.

– Remarks

* If $p(t)$ and $a[n]$ are real-valued, then

$$X(t) =$$

In this case, $f_c = 0$ and $\theta = 0$ lead to baseband PAM.

* $\mathcal{A}$ is a finite set for digital PAM, while it is an infinite set for analog PAM.

* Pulse-code modulation (PCM) is a discrete-time version of digital PAM now with $p[n] = \delta[n]$ and $T = 1$. It is often used to convert an analog audio signal to a digital uncompressed audio signal.

- Quadrature Amplitude Modulation (QAM)
  - Def. QAM generates a real-valued bandpass signal given by

$$X(t) = \mathrm{Re}\left\{ \left( \sum_{n=-\infty}^{\infty} d[n] p(t - nT) \right) e^{\mathrm{j}(2\pi f_c t + \theta)} \right\}$$

    * where
      - $p(t)$ is called the transmit waveform or the transmit pulse,
      - $T$ is often called the symbol period,
      - $1/T$ is called the symbol transmission rate or the baud rate, and
      - $\mathcal{A} \ni a[n]$ is a **two-dimensional constellation with complex-valued constellation points**.

– Remarks

* Analog QAM has $X(t) = m_1(t)\cos(2\pi f_c t) - m_2(t)\sin(2\pi f_c t)$ with $m_1(t)$ and $m_2(t)$ being analog real baseband signals.
* (Pulse-shaped) PSK can be viewed as a special QAM with constellation points being placed on the unit circle.
* **impulse modulator**
* (8-QAM,) 16-QAM, (32-QAM,) 64-QAM, 256-QAM, etc.

* Gray coding vs. natural binary coding

- PSDs
  - Q. Find the PSDs of PAM and QAM signals under the assumption that the data sequence $(d[n])_n$ is WSS with
    - the auto-correlation function $\phi_{dd}[n] \triangleq \mathbb{E}\{d[m+n]d^*[m]\}$
    - and the PSD $\Phi_{dd}(e^{\mathrm{j}2\pi f}) = \mathcal{F}\{\phi_{dd}[n]\} \triangleq \sum_{n=-\infty}^{\infty} \phi_{dd}[n]e^{\mathrm{j}2\pi fn}$.
  - A. (A General Result)
    - The time-averaged auto-correlation function of the complex envelope is given by

$$\begin{aligned}
\mathcal{R}_{X_l X_l}(\tau) &= \ldots \\
&= \mathcal{A}\left\{ \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \phi_{dd}[m-n] p(t+\tau-mT) p^*(t-nT) \right\} \\
&= \mathcal{A}\left\{ \sum_{l=-\infty}^{\infty} \phi_{dd}[l] \sum_{m=-\infty}^{\infty} p(t+\tau-mT) p^*(t-(m-l)T) \right\} \\
&= \sum_{l=-\infty}^{\infty} \phi_{dd}[l] \frac{1}{T} \int_0^T \sum_{m=-\infty}^{\infty} p(t+\tau-mT) p^*(t-(m-l)T) dt
\end{aligned}$$

$$= \frac{1}{T} \sum_{l=-\infty}^{\infty} \phi_{dd}[l] \int_{-\infty}^{\infty} p(t+\tau) p^*(t+lT) dt$$

$$= \frac{1}{T} \sum_{l=-\infty}^{\infty} \phi_{dd}[l] \tilde{p}(\tau - lT).$$

where $\tilde{p}(\tau)$ is **the time-autocorrelation function of the transmit pulse** $p(t)$ defined as

$$\tilde{p}(\tau) \triangleq \int_{-\infty}^{\infty} p(t) p^*(t-\tau) dt = p(\tau) * p^*(-\tau).$$

* Thus, the CTFT $\mathcal{F}\{\mathcal{R}_{X_l X_l}(\tau)\}$ leads to the PSD of the complex envelope given by

$$S_{X_l X_l}(f) = ...$$

$$= \frac{1}{T} \Phi_{dd}(e^{\mathrm{j}2\pi f T}) |P(f)|^2.$$

* Therefore, the PSD of $X(t)$ is given by

$$S_{XX}(f) = \frac{S_{X_l X_l}(f - f_c) + S_{X_l X_l}(-(f + f_c))}{4}$$

– Remarks

* In general, **not only the transmit pulse but also the auto-correlation of the data sequence affect the PSD of $X(t)$.**
  - Recall that the complementary auto-correlation function of the complex envelope does not affect the PSD of the transmitted real bandpass signal.
  - If $(d[n])_n$ has non-zero mean, then $\phi_{dd}[l]$ has a constant term, i.e., $\Phi_{dd}(e^{\mathrm{j}2\pi fT})$ has impulse train with period $1/T$.
  - If $\phi_{dd}[l] = \delta[l]$ and $p(t) = p_T(t)$, then $S_{X_l X_l}(f) = T\mathrm{sinc}^2(fT)$.
  - Practical channel codes does not have $\phi_{dd}[l] \propto \delta[l]$. But many of them approximately have such a property.
  - To better shape the PSD, the transmit pulse is shaped. For example, $\mathrm{sinc}(t/T)$ leads to a brick wall PSD with bandwidth $1/T$ at passband.