Digitally Modulated Signals and Their PSDs: OFDM

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- Review. A digitally modulated transmitted signal $X(t)$ is assumed a **real-valued bandpass** random process.
  - PSD of complex envelope $X_l(t)$ and PSD of $X(t)$
  - PSDs of ASK, PSK, PAM, and QAM signals with WSS symbol sequences
  - PSDs of OQPSK and MSK with i.i.d. antipodal symbols
  - PSD of DS-CDMA with random sequence

- Preview.
  - FSK (frequency shift keying)
  - OFDM (orthogonal frequency-division multiplexing) and Its PSD

- FSK (Frequency Shift Keying)
  - Def. FSK is a special case of angle modulation where a digital message is embedded by **shifting (=abruptly changing) the frequency of the carrier**.
  - In particular, given a real-valued bandpass digitally modulated signal $X(t) = A(t)\cos(\theta(t))$, FSK has a constant envelope $A(t) = A, \forall t$, but the total instantaneous angle $\theta(t)$ is time-varying as

$$\theta(t) = \sum_{n=-\infty}^{\infty} (2\pi f[n]t + \theta[n])p_T(t - nT)$$

    where
    * $p_T(t)$ is again the rectangular pulse with support $[0, T)$ and height 1,
    * $f[n] \in F$ is a real-valued scalar,
    * F is a finite set of distinct frequencies, and
    * $\theta[n] \in [0, 2\pi)$.

– Remarks

* $1/T$ is the rate of change in frequency.
* For T seconds, $\log_2 |F|$ bits are transmitted, where $|\cdot|$ is the cardinality of the set.
* Binary FSK (BFSK) has $|F| = 2$. An M-ary FSK (MFSK) has $|F| = M$.
* If $F = \{f_1, f_2, ..., f_M\}$ with $f_i < f_{i+1}$, then usually $\Delta f \triangleq f_{i+1} - f_i$ is chosen constant for all i. That is, the instantaneous **frequencies are usually spaced equally**.
* Fig.
* The spacing Δf is chosen to orthogonalize the symbols and, at the same time, to minimize the signal bandwidth.

* Q. When $f_i \gg 1$, find the condition on the frequency spacing Δf to have

$$\int_{lT}^{(l+1)T} \mathrm{Re}\left\{e^{\mathrm{j}(2\pi f_m)t+\theta_m)}\right\} \mathrm{Re}\left\{e^{\mathrm{j}(2\pi f_n t+\theta_n)}\right\} dt \approx 0, \forall l, \forall m \neq n.$$

* Q. Find the condition on the frequency spacing Δf to have

$$\frac{1}{T}\int_0^T \left(e^{\mathrm{j}2\pi f_m t}\right)\left(e^{\mathrm{j}2\pi f_n t}\right)^* dt = \delta[m-n].$$

* A. The condition holds iff Δf is an integer multiple of $1/T$. This implies that **the minimum frequency spacing for orthogonal MFSK** is the inverse of the symbol period.

- OFDM (Orthogonal Frequency-Division Multiplexing)
  - There are more than a few signal models for OFDM.
  - (Simplified CT Model) Def. An OFDM signal consists of K symbol-synchronized multiple QAM signals and is given by

$$X(t) = \sum_{k=1}^{K} \operatorname{Re}\left\{ \left(\sum_{n=-\infty}^{\infty} d_k[n] p_{T+\Delta T}(t - n(T+\Delta T)) \right) e^{\mathrm{j}(2\pi(f_1 + \frac{k-1}{T})t + \theta_k)} \right\},$$

    where
    * each QAM signal has a **rectangular transmit pulse** $p_{T+\Delta T}(t)$ with support $[0, T+\Delta T)$ and height 1,
    * $(d_k[n])_n$ is the QAM data sequence of the kth QAM signal,
    * $f_k \triangleq f_1 + (k-1)/T$ is the frequency of the kth **subcarrier**, and
    * $\theta_k \in [0, 2\pi)$.
    * $\Delta T > 0$ is chosen to be greater than or equal to the channel impulse response, so that **the carrier spacing $1/T$ is greater than the symbol rate** $1/(T+\Delta T)$.

– Remarks

* Apparently, an OFDM system employs K local oscillators to generate K synchronized subcarriers tones, which is hard to implement.

* However, for practical implementation, a **single LO** is used to generate a carrier tone and an $N(\geq K)$-point **IFFT** (Inverse Fast Fourier Transform) is performed to effectively generate synchronized subcarrier tones.
* The number $N - K$ is called the number of **null subcarriers**.

* A single data stream is first serial-to-parallel **(S/P)** converted.
* The nth IFFT input consists of K data symbols $(d_k[n])_k$ and $N-K$ zero symbols.
* The output of the IFFT is lengthened by adding a **cyclic prefix (CP)** whose effective length in CT is ΔT.
* The CP-added IFFT output is then parallel-to-serial **(P/S)** converted.

* The complex baseband DT signal is digital-to-analog converted and upconverted to a real bandpass signal.

- Why nulling?
  * Recall that $\sum_{n=-\infty}^{\infty} d[n]p(t-nT_c)$ has the PSD

$$\frac{1}{T_c}\Phi_{dd}(e^{\mathrm{j}2\pi f T_c})|P(f)|^2.$$

  * Null subcarriers make the periodic PSD $\Phi_{dd}(e^{\mathrm{j}2\pi f T_c})$ of the DT signal input to the DAC have a spectral null within each period.

  * The spectral null can be utilized by the DAC (and following analog filter) to prevent each subcarrier from appearing more than once at the output.

– Why CP?

* Q. When $x(t) = \mathrm{Re}\left\{p_{T+\Delta T}(t)e^{\mathrm{j}(2\pi f_k t+\theta_k)}\right\}$, what is the output $y(t)$ on $t \in [\tau_1 + \Delta T, \tau_1 + \Delta T + T)$ to the channel with impulse response

$$h(t) = \sum_{l=1}^{L} \alpha_l \delta(t - \tau_l)$$

where $\tau_1 < ... < \tau_L$ and $\tau_L - \tau_1 < \Delta T$.

* A. It can be shown that

$$y(t) = \mathrm{Re}\left\{\sum_{l=1}^{L} \alpha_l e^{-\mathrm{j}(2\pi f_k \tau_l - \theta_k)} p_{T+\Delta T}(t - \tau_l) e^{\mathrm{j}2\pi f_k t}\right\},$$

which reduces to

$$y(t) = \mathrm{Re}\left\{\left(\sum_{l=1}^{L} \alpha_l e^{\mathrm{j}(2\pi f_k \tau_l + \theta_k)}\right) e^{\mathrm{j}2\pi f_k t}\right\},$$

for $t \in [\tau_1 + \Delta T, \tau_1 + \Delta T + T)$.

* As seen in FSK, for $t \in [\tau_1 + \Delta T, \tau_1 + \Delta T + T)$, the K subcarrier signals are mutually orthogonal.
* Note
  - that the transmitted signals are not orthognal,
  - that the received signals are not orthogonal, but
  - that, thanks to the existence of the CP, the suboptimal receiver can orthogonalize the signals on the subcarriers.

- PSD
  - Q. Find the PSD of an OFDM transmitted signal under the assumption that the data sequences $(d_k[n])_n$ are uncorrelated and each sequence is white.
  - A.
    * Each subcarrier has a QAM signal with rectangular pulse. $\Rightarrow$ The PSD of each subcarrier is a sinc squared shape.
    * Moreover, subcarrier signals are uncorrelated. $\Rightarrow$ Total PSD is the sum of sinc squares shifted by f_k.
    * Thus, $S_{XX}(f) = S_{X_lX_l}(f - f_1)/4 + S_{X_lX_l}(-(f + f_1))/4$, where

$$S_{X_lX_l}(f) = (T + \Delta T) \sum_{k=1}^{K} \mathrm{sinc}^2\left(\left(f - \frac{k-1}{T} \right) (T + \Delta T) \right).$$

  - Note that

$$\sum_{n=-\infty}^{\infty} \mathrm{sinc}^2(f - n) = 1, \forall n.$$