Digital Baseband/IF Modem: DAC and Uniform Interpolator

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- Review.
  - Modulation, Demodulation, and PSD
  - Digital BB/IF Modem
  - ADC and uniform sampling
  - Sampling viewed in FD
  - The Nyquist sampling theorem

- Preview.
  - DAC and uniform interpolation
  - Interpolation viewed in FD
  - Proof of the Nyquist sampling theorem

- Modem
  - A **modem** consists of a **mod**ulator part and a **dem**odulator part to enable two-way communications.
  - A modem consists of a **digital** system part and an **analog** system part.

- Digital Baseband Modem vs. Digital IF Modem
  - If two DACs generate two CT baseband signals (I and Q) and two ADCs admit two CT baseband signals (I and Q),
  - then the modem is called a **digital baseband modem**.

  - If one DAC generates a CT IF signal and one ADC admits a CT IF signal,
  - then the modem is called a **digital IF modem**.

- Two Key Words
  - sampling vs. interpolation
  - time domain vs. frequency domain
- Uniform Interpolator and DAC
  - Def. A **uniform interpolator with**
    - **interpolation function** $p(t)$ and **interpolation rate** $1/T_I$

    is a linear system with
    - a **DT input signal** $y_d[n]$ and a **CT output signal** $y(t)$,

    where the I/O relation is given by

$$y(t) = \sum_{n-=\infty}^{\infty} y_d[n]p(t - nT_I).$$

  - Remarks
    - Different interpolation functions generate different CT signals.
    - Different rates generate different CT signals.

* $p(t)$
  - is called the **interpolation pulse/waveform**, and
  - is the impulse response of the **interpolation filter** following an impulse modulator.
  - ZOH (zero-order hold), FOH (first-order hold)
* $1/T_I$
  - is often called the **conversion rate** or, misleadingly, sampling rate, and
  - has the unit 'samples per second' denoted by [S/s].
* A **DAC** is a real-world approximation of a uniform interpolator with a real-valued input, a real-valued interpolation function, and a real-valued output.
  - A real-world DAC often **upsamples the input signal and then interpolates**. We will see later why this gives an advantage.
  - Each sample consists of a finite number of bits.
* A uniform interpolator looks similar to a QAM modulator with Tx pulse $p(t)$ and symbol Tx rate $1/T_I$.

- Interpolation viewed in FD
  - Q. Find the CTFT $Y(f)$ of the output in terms of the conversion rate $1/T_I$, the DTFT $Y_d(e^{j2\pi f})$ of the input, and the CTFT $P(f)$ of the interpolation function.
  - A.

$$
\begin{aligned}
Y(f) &\triangleq \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}\,\mathrm{d}t \\
&= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} y_d[n]p(t-nT_I)e^{-j2\pi ft}\,\mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} y_d[n] \int_{-\infty}^{\infty} p(t-nT_I)e^{-j2\pi ft}\,\mathrm{d}t \\
&= \sum_{n=-\infty}^{\infty} y_d[n]P(f)e^{-j2\pi fnT_I} = \left(\sum_{n=-\infty}^{\infty} y_d[n]e^{-j2\pi fT_I n}\right) P(f) \\
&= Y_d(e^{j2\pi fT_I})P(f).
\end{aligned}
$$

– Remarks

* The CTFT of the output factors into the **scaled DTFT** $Y_d(e^{j2\pi fT_I})$ of the input and the **CTFT** $P(f)$ of the interpolation function.

* The DTFT $Y_d(e^{j2\pi f})$ has period 1.
  · The DTFT $Y_d(e^{j2\pi fT_I})$ has period the same as the conversion rate $1/T_I$.

* The CTFT of $y(t)$ is proportional to the CTFT of $p(t)$.
  · The ESD of $y(t)$ is given by $|Y_d(e^{j2\pi fT_I})|^2 \cdot |P(f)|^2$.
  · Recall that the complex-envelope of a QAM signal $x_l(t) = \sum_{-\infty}^{\infty} d[n]p(t-nT)$ has the PSD

$$S_{X_lX_l}(f) = \frac{1}{T}\Phi_{dd}(e^{j2\pi fT})|P(f)|^2.$$

– **Nyquist's Sampling Theorem**.

Suppose that a CT signal $x(t)$ is **finite-energy and bandlimited to** $f \in [-W, W]$, i.e.,

$$E_{xx} = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-W}^{W} |X(f)|^2 df.$$

Then, **the uniform sampling** $x_d[n] \triangleq x(nT_s + \Delta)$ **preserves all the information on** $x(t)$ **iff**

$$\frac{1}{T_s} \text{ [samples/second]} \geq 2W \text{ [hertz]}$$

i.e., the sampling rate, whose unit often denoted by **[S/s]**, is greater than or equal to twice the bandwidth in [Hz].

– A **proof by construction** can be found as follows.

– Proof. It suffices to prove that, when $1/T_s = 2W$, $x(t)$ can be perfectly recovered from its samples.

* Choose $T = T_s = T_I$. Then, the CT signal $x(t)$ is sampled as $x_d[n] = x(nT + \Delta), \forall n$. WLOG, we set $\Delta = 0$.
  As shown already, we have

$$X_d(e^{j2\pi f}) = \frac{1}{T} \sum_{m=-\infty}^{\infty} X\left(\frac{f-m}{T}\right).$$

* Now, consider interpolating $x_d[n]$ with $p(t) = \mathrm{sinc}(t/T)$ as $y(t) = \sum_{n=-\infty}^{\infty} x_d[n]\mathrm{sinc}((t - nT)/T)$.
  Then, since

$$P(f) = \begin{cases} T & \text{for } |f| \le 1/(2T) = W \\ 0 & \text{elsewhere,} \end{cases}$$

  we have

$$Y(f) = X_d(e^{j2\pi fT})P(f) = \left\{\frac{1}{T} \sum_{m=-\infty}^{\infty} X\left(f - \frac{m}{T}\right)\right\} P(f) = X(f).$$

* Therefore, $x(t)$ can be perfectly recovered from its samples uniformly taken at the rate $2W$.

– Remarks

* If $1/T \geq 2W$, then $P(f)$ such that

$$P(f) = \begin{cases} T, & \text{for } \quad |f| \leq W \\ \text{arbitrary}, & \text{for } W < |f| < 1/T - W \\ 0, & \text{for } \quad |f| \geq 1/T - W \end{cases}$$

  can also perfectly recover $y(t)$.

* If the interpolation filter with $P(f)$ does not satisfy the above condition, then the DAC needs to be followed by a CT filter with $Q(f)$ such that
  the overall response $P(f)Q(f)$ satisfies the above condition.
  · ZOH: inverse sinc
  · FOH: inverse sinc square
* Delay is usually tolerated.