Digital Baseband/IF Modem: Downsampling and Decimation

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- Review.
  - Modulation, Demodulation, and PSD
  - Digital BB/IF Modem
  - ADC, DAC, and the Nyquist Sampling Theorem
  - Upsampling and Interpolation Viewed in TD and FD
  - Tx Part of Digital BB/IF Modem

- Preview.
  - Downsampling and Decimation Viewed in TD and FD
  - Rx Part of Digital BB Modem
  - Rx Part of Digital IF Modem

- Downsampling
  - Def. **Downsampling by a factor of** M is a DT-to-DT linear operation with
    * a **DT input signal** $x_d[m]$ and
    * a **DT output signal** $y_d[n]$,

    where the I/O relation is given by

$$y_d[n] \triangleq x_d[nM + \Delta], \forall n.$$

  - Fig. $\downarrow M$
  - Remarks
    * Different rates generate different output signals.
    * Different offsets generate different output signals.

* Notice the similarity to CT-to-DT uniform sampling given by $x_d[n] = x(nT_s + \Delta)$.
* Note that
  - If $x_d[m]$ is obtained by sampling a CT signal $x(t)$ at rate $1/T_s$, i.e., $x_d[m] = x(mT_s) \Longleftrightarrow$

$$X_d(e^{\mathrm{j}2\pi f}) = \frac{1}{T_s} \sum_{m=-\infty}^{\infty} X\left(\frac{f-m}{T_s}\right), \tag{1}$$

  - then $y_d[n]$ corresponds to the DT signal obtained by sampling $x(t)$ M times slowly at rate $1/(MT_s)$, i.e., $y_d[n] = x(nMT_s) \Longleftrightarrow$

$$Y_d(e^{\mathrm{j}2\pi f}) = \frac{1}{MT_s} \sum_{m=-\infty}^{\infty} X\left(\frac{f-m}{MT_s}\right), \tag{2}$$

  - Thus, a downsampler can be used to reduce the sampling rate by M.
  - Since downsampling expands the bandwidth, **aliasing** may occur.

- Q. Find the DTFT of $y_d[n]$ in terms of the downsampling factor M and the DTFT of $x_d[m]$.
- A. By comparing Eqs. (1) and (2), we can guess

$$Y_d(e^{j2\pi f}) = \frac{1}{M} \sum_{m=\langle M \rangle} X_d(e^{j2\pi \frac{f-m}{M}}).$$

  The derivation is omitted.
- Remarks
  * The output DTFT factors into the product of
    · the **sampling rate** $1/M$ and
    · the **periodic spectrum** $\sum_{m=\langle M \rangle} X_d(e^{j2\pi \frac{f-m}{M}})$.
  * The input DTFT is periodic with period 1.
    · First, a folding is performed with only M terms. Each term is shifted by a multiple of $1/M$.
    · Then, horizontal scale is changed to make $1/M$ to 1.
  * During the folding,
    · **aliasing** may occur and
    · spectrum expands to higher frequency.

– Q. Find the necessary and sufficient condition on downsampling rate for perfect reconstruction, when $x_d[m]$ is bandlimited to $|f| \leq W \leq 1/2$.

– A. (Nyquist sampling theorem)

$$\frac{1}{M} \geq 2W.$$

– Proof.

* Consider cascading a downsampler by M, an upsampler by M, and an ideal LPF with impulse response $\mathrm{sinc}(n/M)$.
* Let $x_d[m], y_d[n], z_d[m]$ be the DT signals.
* Then,

$$Y_d(e^{\mathrm{j}2\pi f}) = \frac{1}{M} \sum_{m=\langle M \rangle} X_d(e^{\mathrm{j}2\pi \frac{f-m}{M}})$$

and

$$Z_d(e^{\mathrm{j}2\pi f}) = Y_d(e^{\mathrm{j}2\pi f M}) R(e^{\mathrm{j}2\pi f}).$$

* Thus, $Z_d(e^{\mathrm{j}2\pi f}) = X_d(e^{\mathrm{j}2\pi f})$ iff the condition holds.
* Therefore, the conclusion follows.

– Def. **Decimation by a factor of** M is a DT-to-DT linear operation having

  * a **DT LPF with cutoff frequency** $1/(2M)$ as an **anti-aliasing filter** and
  * a following M **times downsampler**.
  * Fig.

– Remarks

  * A **decimator** can be used for **sampling rate reduction by** M.
  * Decimation means verbatimly killing one out of ten.
  * Often, decimation means downsampling that does not use an AAF.

- Motivations Revisited
  - Rx Part of Digital BB Modem
    - The real BB signals sampled by the ADCs at rate M/T are given after combining by

$$x_{l,d}[n] = x_l(nT/M) = \sum_{m=-\infty}^{\infty} d[m]p_d[n - mM]$$

      with $p_d[n] = p(nT/M)$ being the M-times oversampled version of $p(t)$.
    - First, matched filtering is often performed on this signal to acquire timing and estimate channel response.
    - Then, decimation is performed with a proper decimation factor and a proper offset.
    - The AAF removes adjacent channel interference (ACI) and out-of-band AWGN.

– Rx Part of Digital IF Modem

* The IF signal $x_{\mathrm{IF}}(t)$ sampled by the ADC at rate LM/T is given in TD by

$$x_{\mathrm{IF},d}[n] = \mathrm{Re}\left\{ \sum_{m=-\infty}^{\infty} d[m] p\left(\frac{nT}{LM} - mT \right) e^{\mathrm{j}2\pi f_{\mathrm{IF}} \frac{nT}{LM}} \right\}$$

and in FD by

* First, this signal is down-converted to complex baseband in TD as

$$x_{\mathrm{IF},d}[n] \times 2e^{-\mathrm{j}2\pi f_{\mathrm{IF}} \frac{nT}{LM}} \tag{3}$$

and in FD as

* Note that this signal is NOT equal to $x_l(nT/(LM))$, whose bandwidth is less than or equal to $1/(2L)$.
* Moreover, it may contain ACI and bandlimited AWGN.

* Second, decimation by a factor of L is performed.
  - Direct downsampling of Eq. (3) incurs aliasing.
  - Thus, an AAF must precede the downsampler.
  - The AAF also removes ACI and out-of-band AWGN.
* Now, we have $x_l(nT/M)$ corrupted by bandlimited AWGN.
* Third, matched filtering is often performed on the oversampled signal to acquire timing and estimate channel response.
* Fourth, decimation is performed once more with a proper decimation factor and a proper offset.