Fourier Analysis 1

Prof. Joon Ho Cho
Department of Electrical Engineering, POSTECH

- Continuous-Time Fourier Transform (CTFT)
  - Definition. Given a function $x(t)$ of time $t \in \mathbb{R}$ in seconds (sec), its Fourier transform denoted by $\mathcal{F}\{x(t)\}$ is defined as

$$\mathcal{F}\{x(t)\} \triangleq \int_{-\infty}^{\infty} x(t)e^{-\mathrm{j}2\pi ft}dt,$$

    where $f \in \mathbb{R}$ is in Hertz (Hz).

  - Note that $\mathcal{F}\{x(t)\}$, if exists, is a function only of frequency f.

– Example. $x(t) = 1$ for $|t| \leq T/2$ and $x(t) = 0$ elsewhere. Then, $\mathcal{F}\{x(t)\}$ exists and is given by

$$\mathcal{F}\{x(t)\} = T\text{sinc}(fT).$$

– Definition. The sinc function denoted by $\text{sinc}(x)$ is defined as

$$\text{sinc}(x) \triangleq \begin{cases} 1, & \text{for } x = 0, \\ \frac{\sin(\pi x)}{\pi x}, & \text{elsewhere.} \end{cases}$$

– Definition. Given a function $X(f)$ of frequency $f \in \mathbb{R}$ in Hz, its inverse Fourier transform denoted by $\mathcal{F}^{-1}\{X(f)\}$ is defined as

$$\mathcal{F}^{-1}\{X(f)\} \triangleq \int_{-\infty}^{\infty} X(f) e^{\mathrm{j}2\pi f t} df,$$

where $t \in \mathbb{R}$ is in sec.

– Note that $\mathcal{F}^{-1}\{X(f)\}$, if exists, is a function only of time t.

– Example. $X(f) = 1$ for $|t| \leq 1/(2T)$ and $X(f) = 0$ elsewhere. Then, $\mathcal{F}^{-1}\{X(f)\}$ exists and is given by

$$\mathcal{F}^{-1}\{X(f)\} = \frac{1}{T}\,\mathrm{sinc}\left(\frac{t}{T}\right).$$

– Theorem. Suppose that $\mathcal{F}\{x(t)\}$ exists for a given continuous-time (CT) signal, i.e.,

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-\mathrm{j}2\pi ft}dt. \tag{1}$$

Then, under certain conditions, $\mathcal{F}^{-1}\{X(f)\}$ exists and becomes equal to $x(t)$, i.e.,

$$x(t) = \int_{-\infty}^{\infty} X(f)e^{\mathrm{j}2\pi ft}df. \tag{2}$$

– Eq. (1) is called the analysis equation because it finds the coefficient $X(f)$ of $e^{\mathrm{j}2\pi ft}$ that is a component of $x(t)$.

– Eq. (2) is called the synthesis equation because it synthesizes $x(t)$ as a weighted integral of $e^{\mathrm{j}2\pi ft}$ for $f \in \mathbb{R}$.

– If $X(f) = \mathcal{F}\{x(t)\}$ and $x(t) = \mathcal{F}^{-1}\{X(f)\}$, then $x(t)$ and $X(f)$ are a Fourier transform pair and this relation is denoted by $x(t) \xleftrightarrow{\mathcal{F}} X(f)$.

– Example. $\delta(t) \xleftrightarrow{\mathcal{F}} 1$ because of the sifting property

$$\int_{-\infty}^{\infty} \delta(t - t_0) g(t) dt = g(t_0).$$

– Example. $e^{j2\pi f_0 t} \xleftrightarrow{\mathcal{F}} \delta(f - f_0)$

- Properties of CTFT
  - Suppose that $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$. Then,

    $x(-t) \overset{\mathcal{F}}{\longleftrightarrow} X(-f)$: Reverse in time corresponds to reverse in frequency.

    $x^*(t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(-f)$: Conjugation in time corresponds to conjugation and reverse in frequency.

    $x^*(-t) \overset{\mathcal{F}}{\longleftrightarrow} X^*(f)$: Conjugation and reverse in time corresponds to conjugation in frequency.

  - Example. $x(t) = Ae^{\mathrm{j}2\pi f_0 t}$

- Properties of CTFT
  - Suppose that $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)$ and that $x(t)$ is real-valued, i.e., $x(t) = x^*(t)$.

    Then, the CTFT $X(f)$ is conjugate symmetrical, i.e.,

$$X(f) = X^*(-f), \forall f.$$

Thus, the magnitude $|X(f)|$ is an even function, and

the phase $\measuredangle X(f)$ is an odd function.

– Suppose that $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)$ and
that $x(t)$ is real and even, i.e., $x(t) = x^*(t) = x(-t)$.
Then, $X(f)$ is real and even, i.e.,

$$X(f) = X^*(f) = X(-f), \forall f.$$

– Example. $x(t) = \operatorname{sinc}\left(\frac{t}{T}\right)$

- Linearity of CTFT
  - Suppose that $x_1(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_1(f)$ and $x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X_2(f)$.
    Then, $a_1 x_1(t) + a_2 X_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} a_1 X_1(f) + a_2 X_2(f)$.

- Time delay corresponds to linear phase shift.
  - Suppose that $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)$.
    Then, $x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)e^{-\mathrm{j}2\pi f t_0}$.

- Modulation corresponds to frequency shift.
  - Suppose that $x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(f)$. Then, $x(t)e^{\mathrm{j}2\pi f_0 t} \overset{\mathcal{F}}{\longleftrightarrow} X(f - f_0)$.

  - Example. Find the CTFT of $x(t) = \mathrm{Re}\left\{b[n]\mathrm{sinc}\left(\frac{t-nT}{T}\right) e^{\mathrm{j}2\pi f_0 t}\right\}$.