Fourier Analysis 2

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- CT Signal-Input CT Signal-Output System
  - Definition. A CT system is a triplet $(\mathcal{X}, \mathcal{M}, \mathcal{Y})$, where
    $\mathcal{X}$ is the set of all admissible input CT signals,
    $\mathcal{M}$ is a mapping rule, and
    $\mathcal{Y}$ is the set of all possible output CT signals.
  - Often, the mapping rule $\mathcal{M}$ is called the CT system.
  - When an input signal $x(t)$ is mapped by the CT system $(\mathcal{X}, \mathcal{M}, \mathcal{Y})$ to $y(t)$, it is denoted by
    $x(t) \longrightarrow y(t)$,
    $x(t) \longrightarrow \mathcal{M}\{x(t)\}$, or
    $y(t) = \mathcal{M}\{x(t)\}$.

- CT Linear System
  - Definition. A CT linear system is a CT system having a mapping rule $\mathcal{L}$ that satisfies the linearity property, i.e.,

$$x_1(t) \longrightarrow y_1(t) \text{ and } x_2(t) \longrightarrow y_2(t)$$

implies

$$a_1x_1(t) + a_2x_2(t) \longrightarrow a_1y_1(t) + a_2y_2(t).$$

In other words,

$$\mathcal{L}\{a_1x_1(t) + a_2x_2(t)\} = a_1y_1(t) + a_2y_2(t).$$

  - Example. $\mathcal{M}\{x(t)\} = |x(t)|$ is not a CT linear system.
  - Example. $\mathcal{L}\{x(t)\} = x(t)e^{\mathrm{j}2\pi f_0 t}$ is a CT linear system.

- CT Time-Invariant (TI) System
  - Definition. A CT TI system is a CT system having a mapping rule $\mathcal{M}$ that satisfies the time-invariance property, i.e.,
$$x(t) \longrightarrow y(t)$$
    implies
$$x(t-t_0) \longrightarrow y(t-t_0), \forall t_0.$$
    In other words,
$$\mathcal{M}\{x(t-t_0)\} = y(t-t_0), \forall t_0.$$
  - Example. $\mathcal{M}\{x(t)\} = |x(t)|$ is a CT TI system.
  - Example. $\mathcal{L}\{x(t)\} = x(t)e^{\mathrm{j}2\pi f_0 t}$ is not a CT TI system for $f_0 \neq 0$.

- CT Linear Time-Invariant (LTI) System
  - Definition. A CT LTI system is a CT system whose mapping rule $\mathcal{H}$ satisfies the linearity and the time-invariance properties.

  - Example. $\mathcal{L}\{x(t)\} = x(t)e^{\mathrm{j}2\pi f_0 t}$ is not a CT LTI system for $f_0 \neq 0$.
  - Example. $\mathcal{H}\{x(t)\} = \sum_{k=0}^{K} a_k \frac{d^k x(t)}{dt^k}$ is a CT LTI system when zero-input zero-output condition holds.

- Impulse Response of a CT LTI System
  - Theorem. Suppose that $(\mathcal{X}, \mathcal{H}, \mathcal{Y})$ is a CT LTI system and $x(t) \longrightarrow y(t)$. Then, there exists a CT signal $h(t)$ such that

$$y(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau.$$

    In particular, $h(t)$ is the response of the system at time t to the impulse applied at time 0, i.e.,

$$h(t) \triangleq \mathcal{H}\{\delta(t)\}.$$

  - Definition. Given two CT signals $h(t)$ and $x(t)$,

$$\int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau$$

    is called the convolution integral of $h(t)$ and $x(t)$ and denoted by $h(t) * x(t)$.

  - $h(t) * x(t)$ is a function of time t.

- Properties of Convolution Integral
  - Commutativity. $h(t) * x(t) = x(t) * h(t)$
  - Distributivity. $\{h_1(t) + h_2(t)\} * x(t) = h_1(t) * x(t) + h_2(t) * x(t)$
  - Associativity. $h_2(t) * \{h_1(t) * x(t)\} = \{h_2(t) * h_1(t)\} * x(t)$

- Order doesn't matter for serially interconnected LTI systems. $h_2(t) * \{h_1(t) * x(t)\} = h_1(t) * \{h_2(t) * x(t)\}$

- (Back to CTFT) Properties of CTFT
  - Convolution corresponds to multiplication.
    * Suppose that $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)$, $h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} H(f)$, and $y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f)$.
      Then, $y(t) = h(t) * x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = H(f)X(f)$.
    * Specialize! Delay corresponds to linear phase shift.
    * Lowpass filter (LPF), ideal LPF
    * Bandpass filter (BPF), ideal BPF

- Multiplication corresponds to convolution.
  * Suppose that $x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(f)$, $c(t) \stackrel{\mathcal{F}}{\longleftrightarrow} C(f)$, and $y(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f)$.
    Then, $y(t) = c(t)x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} Y(f) = C(f) * X(f)$.

  * Specialize! Modulation corresponds to frequency shift.
  * $\mathrm{LPF}\{m(t)\cos(2\pi f_c t + \theta)\cos(2\pi f_c t + \phi)\} = m(t)\cos(\theta - \phi)/2$

– Parseval's Relation and Energy Spectral Density (ESD)

* Theorem.

$$\int_{-\infty}^{\infty} x^*(t)y(t)dt = \int_{-\infty}^{\infty} X^*(f)Y(f)df$$

i.e., inner product in time equals inner product in frequency.

* Corollary.

$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df$$

i.e., the energy of $x(t)$ can be obtained by integrating the ESD $|X(f)|^2$.

- Continuous-Time Fourier Series (CTFS)
  - Theorem. Given a periodic signal $x(t)$ with period T, i.e., $x(t) = x(t+T), \forall t$, under certain conditions, a_k defined as

$$a_k \triangleq \frac{1}{T}\int_{\langle T \rangle} x(t)e^{-\mathrm{j}2\pi\frac{k}{T}t}dt, \forall k \in \mathbb{Z}$$

    satisfies

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{\mathrm{j}2\pi\frac{k}{T}t}. \tag{1}$$

  - The right side of Eq. (1) is called the CTFS representation of $x(t)$.

– CTFS and CTFT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{\mathrm{j}2\pi \frac{k}{T} t} \overset{\mathcal{F}}{\longleftrightarrow} X(f) = \sum_{k=-\infty}^{\infty} a_k \delta\left(f - \frac{k}{T}\right)$$

* LHS. A periodic signal in time consists of complex exponential functions with harmonically related frequencies k/T.
* RHS. The CTFT of a periodic signal consists of uniformly spaced delta functions in the frequency domain, where the spacing corresponds to the inverse $1/T$ of the period.
* Example. An impulse train in time domain is an impulse train in frequency domain. In particular,

$$\sum_{n=-\infty}^{\infty} \delta(t - nT) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{T} \sum_{n=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right).$$

- Discrete-Time Fourier Transform (DTFT)
  - Definition. Given a two-sided sequence $x[n]$ of time $n \in \mathbb{Z}$, its Fourier transform denoted by $\mathcal{F}\{x[n]\}$ is defined as

$$\mathcal{F}\{x[n]\} \triangleq \sum_{n=-\infty}^{\infty} x[n]e^{-\mathrm{j}2\pi fn},$$

    where $f \in \mathbb{R}$ is in Hertz (Hz).

  - Note that $\mathcal{F}\{x[n]\}$, if exists, is a function only of frequency f.
  - $\mathcal{F}\{x[n]\}$ is periodic with period 1.

– If there is no confusion from the CTFT of $x(t)$, then the DTFT of $x[n]$ is denoted by $X(f)$.

– To avoid confusion, the DTFT of $x[n]$ is often denoted by $X(e^{\mathrm{j}2\pi f})$ or $X_d(f)$.

– Definition. Given a periodic function $X(e^{\mathrm{j}2\pi f})$ of frequency $f \in \mathbb{R}$ with period 1, its inverse Fourier transform denoted by $\mathcal{F}^{-1}\{X(e^{\mathrm{j}2\pi f})\}$ is defined as

$$\mathcal{F}^{-1}\{X(e^{\mathrm{j}2\pi f})\} \triangleq \int_{\langle 1 \rangle} X(e^{\mathrm{j}2\pi f})e^{\mathrm{j}2\pi fn}df,$$

with $n \in \mathbb{Z}$.

– Note that $\mathcal{F}^{-1}\{X(e^{\mathrm{j}2\pi f})\}$, if exists, is a function only of time n.

– Theorem. Suppose that $\mathcal{F}\{x[n]\}$ exists for a given discrete-time (DT) signal, i.e.,

$$X(e^{\mathrm{j}2\pi f}) = \sum_{n=-\infty}^{\infty} x[n]e^{-\mathrm{j}2\pi f n}. \qquad (2)$$

Then, under certain conditions, $\mathcal{F}^{-1}\{X(e^{\mathrm{j}2\pi f})\}$ exists and becomes equal to $x[n]$, i.e.,

$$x[n] = \int_{\langle 1\rangle} X(e^{\mathrm{j}2\pi f})e^{\mathrm{j}2\pi f n}df. \qquad (3)$$

– Eq. (2) is called the analysis equation because it finds the coefficient $X(e^{\mathrm{j}2\pi f})$ of $e^{\mathrm{j}2\pi f n}$ that is a component of $x[n]$.

– Eq. (3) is called the synthesis equation because it synthesizes $x[n]$ as a weighted integral of $e^{\mathrm{j}2\pi f n}$ for $f \in \langle 1\rangle$.

- Properties of DTFT
  - ...
  - $x[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{\mathrm{j}2\pi f}) = \frac{1}{N}\sum_{k=-\infty}^{\infty} \delta(f-k/N)$