Complex Baseband Representation of Real-Valued Bandpass LTI Systems: Derivation

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- Big Picture. 3 elements of communication systems
  - Tx
  - Channel: $\leftarrow$ We are here.
  - Rx

- Review. Often, a wireless channel can be well modeled by a real-valued bandpass LTI system. The key is to understand there are **two cases** to consider.

- Preview. In the first case, we use **the complex envelope** of the impulse response of the real-valued bandpass LTI channel. In the second case, we use **the complex baseband equivalent** to the impulse response of the real-valued bandpass LTI channel.

- Q. Given

$$y(t) = h(t) * x(t),$$

  - $x(t)$ is a **real-valued bandpass** signal having the complex envelope $x_l(t)$,
  - $y(t)$ is a **real-valued bandpass** signal having the complex envelope $y_l(t)$, and
  - $h(t)$ is real-valued impulse response of the channel.

  Find the relation between $x_l(t)$ and $y_l(t)$.

- Q. The channel with input $x_l(t)$ and output $y_l(t)$ is an LTI system? If so, what is the impulse response $c(t)$?

- A. The channel is LTI. Thus,

$$y_l(t) = c(t) * x_l(t).$$

There are two cases.

  - Case 1: If $h(t)$ is a **real-valued bandpass** signal with the same bandwidth as $x(t)$, then

$$c(t) = \frac{1}{2} h_l(t)$$

where $h_l(t)$ is **the complex envelope** of $h(t)$.

– Case 2: If **the channel is specular** and its impulse response $h(t)$ is given by

$$h(t) = \sum_{l=1}^{L} \alpha_l \delta(t - \tau_l),$$

then

$$c(t) = \sum_{l=1}^{L} \alpha_l e^{-\mathrm{j}2\pi f \tau_l} \delta(t - \tau_l),$$

where $2c(t)$ can be called **the complex baseband equivalent** to $h(t)$.

- Derivation
  - Case 1: Where does $\frac{1}{2}$ come from?

    Let's examine the channel output $y(t)$ as

$$\begin{aligned} y(t) &= h(t) * x(t) \\ &= \mathrm{Re}\Big\{h_l(t)e^{\mathrm{j}2\pi f_c t}\Big\} * \mathrm{Re}\Big\{x_l(t)e^{\mathrm{j}2\pi f_c t}\Big\} \\ &= \frac{h_l(t)e^{\mathrm{j}2\pi f_c t} + h_l^*(t)e^{-\mathrm{j}2\pi f_c t}}{2} * \frac{x_l(t)e^{\mathrm{j}2\pi f_c t} + x_l^*(t)e^{-\mathrm{j}2\pi f_c t}}{2}. \end{aligned}$$

    There are four terms, where $\left(\cdot\, e^{\mathrm{j}2\pi f_c t}\right) * \left(\cdot\, e^{-\mathrm{j}2\pi f_c t}\right) = 0$ because $\left(\cdot\, e^{\mathrm{j}2\pi f_c t}\right)$ is the pre-envelope that has support only for $f > 0$.
    Fig.

Thus, only two terms remain to give

$$
\begin{aligned}
y(t) &= \frac{1}{2} \cdot \frac{(h_l(t) * x_l(t))e^{\mathrm{j}2\pi f_c t} + (h_l(t) * x_l(t))^* e^{-\mathrm{j}2\pi f_c t}}{2} \\
&= \mathrm{Re}\left\{ \left(\frac{1}{2} h_l(t) * x_l(t) \right) e^{\mathrm{j}2\pi f_c t} \right\}
\end{aligned}
$$

$$
\therefore y_l(t) = \frac{1}{2} h_l(t) * x_l(t)
$$

.

* Implication: This simple relation allows us

1. to ignore any frequency translation encountered in the modulation of a signal.
2. to deal only with the equivalent lowpass signals through equivalent lowpass systems.

– Case 2: Why don't we have 1/2? Why do we have $e^{-\mathrm{j}2\pi f\tau_l}$?

Again, let's examine the channel output $y(t)$ as

$$\begin{aligned}
y(t) &= h(t) * x(t) = \sum_{l=1}^{L} \alpha_l x(t-\tau_l) \\
&= \mathrm{Re}\left\{ x_l(t-\tau_l) e^{\mathrm{j}2\pi f_c(t-\tau_l)} \right\} \\
&= \mathrm{Re}\left\{ \sum_{l=1}^{L} \alpha_l x_l(t-\tau_l) e^{\mathrm{j}2\pi f_c(t-\tau_l)} \right\} \\
&= \mathrm{Re}\left\{ \left[\sum_{l=1}^{L} \alpha_l e^{-\mathrm{j}2\pi f\tau_l} x_l(t-\tau_l) \right] e^{\mathrm{j}2\pi f_c t} \right\} \\
&= \mathrm{Re}\left\{ \left[\left(\sum_{l=1}^{L} \alpha_l e^{-\mathrm{j}2\pi f\tau_l} \delta(t-\tau_l) \right) * x_l(t) \right] e^{\mathrm{j}2\pi f_c t} \right\}.
\end{aligned}$$

Thus, $c(t) \triangleq \sum_{l=1}^{L} \alpha_l e^{-\mathrm{j}2\pi f\tau_l} \delta(t-\tau_l)$ leads to

$$yl(t) = c(t) * x_l(t).$$

Moreover, setting $2c(t) = h_l(t)$ leads to $y_l(t) = \frac{1}{2}h_l(t) * x_l(t)$.

* Implication: The complex baseband equivalent allows us
  1. to ignore the band unlimitedness of the impulse response of the specular channel.
  2. to clearly see why the path gain becomes a complex number in complex baseband.

- Example
  - Q. When $h(t) = f(t)\cos(2\pi f_c t+\theta)$ and $x(t) = g(t)\cos(2\pi f_c t+\phi)$, find $y(t)$, where $f(t)$ and $g(t)$ are real, and $F(f) = G(f) = 0$ for $|f| > f_c$.
  - A. Since $f(t)\cos(2\pi f_c t+\theta) = \mathrm{Re}\big\{f(t)e^{\mathrm{j}(2\pi f_c t+\theta)}\big\} = \mathrm{Re}\big\{f(t)e^{\mathrm{j}\theta}e^{\mathrm{j}2\pi f_c t}\big\}$, we have $h_l(t) = f(t)e^{\mathrm{j}\theta}$. Similarly, $x_l(t) = g(t)e^{\mathrm{j}\phi}$. Thus,

$$\begin{aligned} y_l(t) &= \frac{1}{2}h_l(t) * x_l(t) \\ &= \frac{1}{2}\big(f(t)e^{\mathrm{j}\theta}\big) * \big(g(t)e^{\mathrm{j}\phi}\big) \\ &= \frac{1}{2}(f(t) * g(t))\, e^{\mathrm{j}(\theta+\phi)} \end{aligned}$$

$$\begin{aligned} \therefore y(t) &= \mathrm{Re}\left\{\frac{1}{2}(f(t) * g(t))\, e^{\mathrm{j}(\theta+\phi)}e^{\mathrm{j}2\pi f_c t}\right\} \\ &= \frac{1}{2}(f(t) * g(t))\cos(2\pi f_c t + \theta + \phi). \end{aligned}$$