Independence among Multiple Random Variables

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- Review. The received signal contains a noise $N(t)$ often modeled by an **additive white Gaussian noise (AWGN)**. To understand it, we started to review a pre-requisite of this course, *Probability, Random Variables, and Random Processes*.
  - CDF and PDF of a random variable
  - joint CDF and joint PDF of multiple random variables
  - how to handle multiple complex-valued random variables by using multiple real-valued random variables

- Preview. We will examine the notion of **statistical independence**.
  - conditional probability
  - statistical independence **among multiple events**
  - statistical independence **among multiple random variables**

- Q. Given a random experiment, what is the conditional probability of the event A given the event B?
- A. **The (conditional) probability of (the event)** A **given (the event)** B, or **the probability of (the event)** A **conditioned on (the event)** B is denoted by $\Pr(A|B)$ and is defined as

$$\Pr(A|B) \triangleq \frac{\Pr(A \cap B)}{\Pr(B)} \tag{1}$$

  when $\Pr(B) \neq 0$.

  – Example. Frequency interpretation of conditional probability.

- Q. Given a random experiment and two events A and B, what does that mean by "the event A is independent of the event B"?
- A. It means
$$\Pr(A|B) = \Pr(A), \tag{2}$$
i.e., the conditioning on the event B does not change the probability of the event A.
  - By substituting (1) into (2), it also means
$$\Pr(A \cap B) = \Pr(A) \Pr(B).$$
  - If $\Pr(A) \neq 0$, then it also means
$$\Pr(B|A) = \Pr(B),$$
i.e., the conditioning on the event A does not change the probability of the event B.

- Example. Given a die rolling experiment with
  A being the event of an even number,
  B being the event of a multiple of 3, and C being the event of a odd number,
  - show that A and B are independent, and
  - show that A and C are dependent.

- Q. Show that the independence of A and B is equivalent to
  - the independence of A and B^c
  - the independence of A^c and B, or
  - the independence of A^c and B^c.
- A.

- Q. Given a random experiment and three events A_1, A_2, and A_3, what does that mean by "The three events are statistically independent."?
- A. It means that
  - an event that is in terms of the events A_i with i in an index set $I(\neq \emptyset) \subset \{1, 2, 3\}$ and
    another event that is in terms of the events A_j with j in the index set $J(\neq \emptyset) \subset I^c$
    are always independent regardless of the choice of I and J.
  - For example,
    * A_i and A_j are independent for $i \neq j$.
    * A_1 and $A_2 \cup A_3$ are independent.
    * A_1 and $A_2 \cup A_3^c$ are independent.
  - This can be compactly stated as
  (a) $\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1) \Pr(A_2) \Pr(A_3)$ and
  (b) any two event are independent, i.e., the pairwise independence.

* Note that
  - **the pairwise independence is NOT equivalent to the independence of the three events,** and
  - $\Pr(A_1 \cap A_2 \cap A_3) = \Pr(A_1)\Pr(A_2)\Pr(A_3)$ **is NOT equivalent to the pairwise independence.**

- Q. Show that the independence among A_1, A_2, and A_3 implies the independence of A_1 and $A_2 \cap A_3^c$.

- A.

  – Other cases can be similarly shown.

- Q. How do we define the statistical independence among N events $A_1, A_2, ..., A_N$?
- A. It means that
  - an event that is in terms of the events A_i with i in an index set $I(\neq \emptyset) \subset \{1, 2, ..., N\}$ and
    another event that is in terms of the events A_j with j in the index set $J(\neq \emptyset) \subset I^c$
    are always independent regardless of the choice of I and J.
  - This can be compactly stated as
    (a) $\Pr(\cap_{n=1}^{N} A_n) = \prod_{n=1}^{N} \Pr(A_n)$ and
    (b) any $n-1$ events with $n \leq N$ are independent.

    * Note that
      - the pairwise independence is NOT equivalent to the independence of the N events, and
      - $\Pr(\cap_{n=1}^{N} A_n) = \prod_{n=1}^{N} \Pr(A_n)$ is NOT equivalent to the independence of the N events.

- Q. What does that mean by "Two random variables X and Y are statistically independent."?
- A. It means that
  - an event that is in terms of the random variable X and another event that is in terms of the random variable Y are always independent events.
  - Note that **the independence of two random variables** is defined based on **the independence of two events**.
- Q. Find an equivalent statement to the above answer now in terms of the joint CDF and the jonit PDF.
- A.
  - The independence between X and Y implies

$$F_{X,Y}(x,y) = ... = F_X(x)F_Y(y), \forall(x,y),$$

  i.e., **the joint CDF factors** into the product of the marginal CDFs.

This implies

$$f_{X,Y}(x,y) = ... = f_X(x)f_Y(y), \forall(x,y),$$

i.e., **the joint PDF factors** into the product of the marginal PDFs.

– Now, we can easily show that

$$f_{X,Y}(x,y) = f_X(x)f_Y(y), \forall(x,y) \tag{3}$$

implies

$$F_{X,Y}(x,y) = F_X(x)F_Y(y), \forall(x,y), \tag{4}$$

i.e., (3) and (4) are equivalent. Moreover, this implies

$$\Pr(\{X \in A\} \cap \{Y \in B\}) = ... = \Pr(\{X \in A\} \Pr(\{Y \in B\}), \forall A, \forall B.$$

– Therefore, the statistical independence of two random variables X and Y is equivalent to (3) or (4).

- Q. What does that mean by "N random variables $X_1, X_2, ...,$ and X_N are statistically independent."?
- A. It means that
  - an event that is in terms of the random variables X_i with i in an index set $I(\neq \emptyset) \subset \{1, 2, ..., N\}$ and
    another event that is in terms of the random variables X_j with j in the index set $J(\neq \emptyset) \subset I^c$
    are always independent regardless of the choice of I and J.
  - This can be compactly stated as

$$F_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_n) = \prod_{n=1}^{N} F_{X_n}(x_n), \forall(x_1, x_2, ..., x_n) \quad (5)$$

    or equivalently as

$$f_{X_1,X_2,...,X_N}(x_1, x_2, ..., x_n) = \prod_{n=1}^{N} f_{X_n}(x_n), \forall(x_1, x_2, ..., x_n), \quad (6)$$

    i.e., the joint CDF factors and the joint PDF factors.

– Q. Why don't we need the statistical independence of any $n-1$ random variables with $n \leq N$?

  * commonality
  * difference

– A. This is because Eq. (5) or Eq. (6) implies the statistical independence of any $n-1$ random variables with $n \leq N$.

- Q. Can you extend the above notion of statistical independence to those among
  – multiple random vectors,
  – multiple random matrices,
  – multiple DT random processes,
  – multiple CT random processes, etc.?
- A.