# *Proper-Complex Gaussian Random Vectors*

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- Review. The received signal contains a noise $N(t)$ often modeled by an **additive white Gaussian noise (AWGN)**. To understand it, we started to review a pre-requisite of this course, *Probability, Random Variables, and Random Processes*.
  - statistical independence of multiple events, and multiple random variables
  - correlation, covariance, and correlation coefficient of two real-valued random variables, and two complex-valued random variables.
  - real-valued Gaussian random vectors
    * characteristic function of a real-valued random variable
    * real-valued Gaussian random vectors

- Preview. We will continue examining the notion of **Gaussianity**.
  - complex-valued Gaussian random vectors
  - proper-complex Gaussian random vectors

- Complex-Valued Gaussian Random Vector
  - Definition. Suppose that $\underline{X} \in \mathbb{R}^N$ and $\underline{Y} \in \mathbb{R}^N$ are real-valued random vectors such that $F_{\underline{X},\underline{Y}}(\underline{x}, \underline{y})$ exists. Then,

$$\underline{Z} \triangleq \underline{X} + \mathrm{j}\underline{Y}$$

is called **a complex-valued random vector** of length $N$.

  * abuse of notation
  * A complex-valued random vector $\underline{Z} \in \mathbb{C}^N$ is completely characterized by the joint CDF/PDF/CF of $\mathrm{Re}\{\underline{Z}\}$ and $\mathrm{Im}\{\underline{Z}\}$.
  * A complex-valued random vector of length $N$ is equivalent to a real-valued random vector of length $2N$.

– Definition.

A random vector $\underline{Z} \triangleq \underline{X} + \mathrm{j}\underline{Y} \in \mathbb{C}^N$
is **a complex-valued Gaussian random vector**
if the real-valued random vectors $\underline{X} \in \mathbb{R}^N$ and $\underline{Y} \in \mathbb{R}^N$
are jointly Gaussian, i.e.,
there exists a length-$2N$ real-valued vector $\underline{\mu}$ and
a size-$2N$ symmetric positive semi-definite real-valued matrix $C$
such that

$$\begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} \sim \mathcal{N}(\underline{\mu}, C).$$

∗ Q. Do we always have to convert a complex Gaussian random vector to a real Gaussian random vector?

– Theorem. A complex Gaussian random vector $\underline{Z} \in \mathbb{C}^N$ is completely characterized by

* the **mean** vector $\underline{\mu}_{\underline{Z}} \triangleq \mathbb{E}[\underline{Z}]$,
* the **covariance** matrix $C_{\underline{ZZ}} \triangleq \mathbb{E}\left[(\underline{Z} - \underline{\mu}_{\underline{Z}})(\underline{Z} - \underline{\mu}_{\underline{Z}})^H\right]$, and
* the **pseudo-covariance** matrix $\tilde{C}_{\underline{ZZ}} \triangleq \mathbb{E}\left[(\underline{Z} - \underline{\mu}_{\underline{Z}})(\underline{Z} - \underline{\mu}_{\underline{Z}})^T\right]$.

– Proof.

* We know that $\underline{Z}$ is completely characterized by the characteristic function $\Phi_{\underline{W}}(\underline{\omega})$ of

$$\underline{W} \triangleq \begin{bmatrix} \underline{X} \\ \underline{Y} \end{bmatrix} \triangleq \begin{bmatrix} \mathrm{Re}\{\underline{Z}\} \\ \mathrm{Im}\{\underline{Z}\} \end{bmatrix} \in \mathbb{R}^{2N}$$

and that $\Phi_{\underline{W}}(\underline{\omega})$ is determined by $\underline{\mu}_{\underline{W}}$ and $C_{\underline{WW}}$. Thus, we are to show that $\underline{\mu}_{\underline{W}}$ and $C_{\underline{WW}}$ are determined by $\underline{\mu}_{\underline{Z}}$, $C_{\underline{ZZ}}$, and $\tilde{C}_{\underline{ZZ}}$.

* First, it can be easily seen that

$$\underline{\mu}_{\underline{W}} = \begin{bmatrix} \mathbb{E}[\underline{X}] \\ \mathbb{E}[\underline{Y}] \end{bmatrix} = \begin{bmatrix} \mathrm{Re}\{\underline{\mu}_{\underline{Z}}\} \\ \mathrm{Im}\{\underline{\mu}_{\underline{Z}}\} \end{bmatrix}$$

is determined by $\underline{\mu}_{\underline{Z}}$.

* Second, note that

$$C_{\underline{WW}} = \begin{bmatrix} C_{\underline{XX}} & C_{\underline{XY}} \\ C_{\underline{YX}} & C_{\underline{YY}} \end{bmatrix}.$$

Since

$$\begin{aligned} C_{\underline{ZZ}} &= \mathbb{E}\left[\left\{(\underline{X} - \underline{\mu}_{\underline{X}}) + \mathrm{j}(\underline{Y} - \underline{\mu}_{\underline{Y}})\right\}\left\{(\underline{X} - \underline{\mu}_{\underline{X}})^T - \mathrm{j}(\underline{Y} - \underline{\mu}_{\underline{Y}})^T\right\}\right] \\ &= (C_{\underline{XX}} + C_{\underline{YY}}) + \mathrm{j}(-C_{\underline{XY}} + C_{\underline{YX}}) \end{aligned}$$

and

$$\begin{aligned} \tilde{C}_{\underline{ZZ}} &= \mathbb{E}\left[\left\{(\underline{X} - \underline{\mu}_{\underline{X}}) + \mathrm{j}(\underline{Y} - \underline{\mu}_{\underline{Y}})\right\}\left\{(\underline{X} - \underline{\mu}_{\underline{X}})^T + \mathrm{j}(\underline{Y} - \underline{\mu}_{\underline{Y}})^T\right\}\right] \\ &= (C_{\underline{XX}} - C_{\underline{YY}}) + \mathrm{j}(C_{\underline{XY}} + C_{\underline{YX}}), \end{aligned}$$

we have

$$
\begin{aligned}
C_{\underline{X}\underline{X}} &= \frac{1}{2}\left[\mathrm{Re}\{C_{\underline{Z}\underline{Z}}\} + \mathrm{Re}\{\tilde{C}_{\underline{Z}\underline{Z}}\}\right] \\
C_{\underline{Y}\underline{Y}} &= \frac{1}{2}\left[\mathrm{Re}\{C_{\underline{Z}\underline{Z}}\} - \mathrm{Re}\{\tilde{C}_{\underline{Z}\underline{Z}}\}\right] \\
C_{\underline{X}\underline{Y}} &= \frac{1}{2}\left[-\,\mathrm{Im}\{C_{\underline{Z}\underline{Z}}\} + \mathrm{Im}\{\tilde{C}_{\underline{Z}\underline{Z}}\}\right] \\
C_{\underline{Y}\underline{X}} &= \frac{1}{2}\left[\mathrm{Im}\{C_{\underline{Z}\underline{Z}}\} + \mathrm{Im}\{\tilde{C}_{\underline{Z}\underline{Z}}\}\right].
\end{aligned}
$$

Thus, $C_{\underline{W}\underline{W}}$ is determined by $C_{\underline{Z}\underline{Z}}$ and $\tilde{C}_{\underline{Z}\underline{Z}}$.

∗ Therefore, the conclusion follows.

– Notation. (Unconventional) If a complex-valued random vector $\underline{Z}$ with mean $\underline{\mu}$, covariance $C$, and pseudo-covariance $\tilde{C}$ is Gaussian, then it is denoted as

$$\underline{Z} \sim \mathcal{CN}(\underline{\mu}; C, \tilde{C}).$$

– Definition. If $\underline{Z} \sim \mathcal{CN}(\underline{\mu}; C, \widetilde{C})$ has a vanishing pseudo-covariance matrix, i.e., $\widetilde{C} = O$, then it is called **proper-complex Gaussian**.

– Notation. (Conventional) $\underline{Z} \sim \mathcal{CN}(\underline{\mu}; C, O)$ is simply denoted as

$$\underline{Z} \sim \mathcal{CN}(\underline{\mu}, C).$$

– Theorem. If $\underline{Z} \sim \mathcal{CN}(\underline{\mu}, C)$ has length $N$ and $C \succ 0$, then

$$\begin{aligned} f_{\underline{Z}}(\underline{z}) &\triangleq \frac{1}{\pi^N \det(C)} \exp\Big(-\big(\underline{Z} - \underline{\mu}\big)^H C^{-1} \big(\underline{Z} - \underline{\mu}\big)\Big) \\ &= f_{\underline{X},\underline{Y}}(\underline{x}, \underline{y}) \end{aligned}$$

where $\underline{Z} \triangleq \underline{X} + \mathrm{j}\underline{Y}$.

– Proof. Omitted.

* Specialize! $N = 1$. If $Z \sim \mathcal{CN}(0, 2\sigma^2)$, then the joint density of $X$ and $Y$ such that $Z = X + \mathrm{j}Y$ is given by

$$f_{X,Y}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left( -\frac{x^2 + y^2}{2\sigma^2} \right)$$

- $X \sim \mathcal{N}(0, \sigma^2)$
- $Y \sim \mathcal{N}(0, \sigma^2)$
- $X$ and $Y$ are independent.
- 3 dB problem
- If non-zero mean, then circularly symmetric joint PDF about the mean.

– Definition. $\underline{Z} \sim \mathcal{CN}(\underline{0}, C)$ is called **a proper-complex white Gaussian random vector** if there exists $\sigma^2$ such that

$$C = 2\sigma^2 I.$$

∗ Note that

$$\begin{aligned}
\mathrm{Cov}[\mathrm{Re}\{Z_i\}, \mathrm{Re}\{Z_j\}] &= \sigma^2 \delta_{i,j} \\
\mathrm{Cov}[\mathrm{Re}\{Z_i\}, \mathrm{Im}\{Z_j\}] &= 0 \\
\mathrm{Cov}[\mathrm{Im}\{Z_i\}, \mathrm{Im}\{Z_j\}] &= \sigma^2 \delta_{i,j}.
\end{aligned}$$

∗ In other words, $\underline{X} \triangleq \mathrm{Re}\{\underline{Z}\}$ and $\underline{Y} \triangleq \mathrm{Im}\{\underline{Z}\}$ are i.i.d. with

$$\mathcal{N}(\underline{0}, \sigma^2 I).$$

∗ Every real and imaginary components are i.i.d. $\mathcal{N}(0, \sigma^2)$.

– Properties of proper-complex Gaussian random vectors

* $\underline{\mu}_{\underline{Z}}$ and $C_{\underline{Z}\underline{Z}}$ completely characterize the distribution.
* Theorem. An affine transform of $\underline{Z}$, e.g.,

$$\underline{W} = A\underline{Z} + \underline{b} \qquad \begin{aligned} A &\in \mathbb{C}^{M\times N} \\ \underline{b} &\in \mathbb{C}^{M\times 1} \end{aligned}$$

  leads to a proper-complex Gaussian random vector such that

$$\underline{W} \sim \mathcal{CN}(A\underline{\mu}_{\underline{Z}} + \underline{b}, AC_{\underline{Z}\underline{Z}}A^H)$$

* Proof.
  · Gaussianity can be justified by considering equivalent real-valued Gaussian random vector.
  · mean vector

  · covariance matrix

  · complementary covariance matrix

- Specialize! Each element of a proper-complex Gaussian random vector is a proper-complex Gaussian random variable.

- Specialize! A weighted sum of $Z_i$'s as $W = \underline{a}^H \underline{Z}$ is a proper-complex Gaussian random variable having a circularly symmetric (about the mean $\underline{a}^H \underline{\mu}_{\underline{Z}}$) PDF having independent real and imaginary parts satisfying

$$\mathrm{Re}\{W\} \sim \mathcal{N}\left(\mathrm{Re}\{\underline{a}^H \underline{\mu}_{\underline{Z}}\}, \frac{1}{2}\underline{a}^H C_{\underline{Z}\,\underline{Z}}\underline{a}\right)$$
$$\mathrm{Im}\{W\} \sim \mathcal{N}\left(\mathrm{Im}\{\underline{a}^H \underline{\mu}_{\underline{Z}}\}, \frac{1}{2}\underline{a}^H C_{\underline{Z}\,\underline{Z}}\underline{a}\right).$$

- Any collection of the elements of a proper-complex Gaussian random vector is a proper-complex Gaussian random vector.

– Example 1. When $\underline{Z} \sim \mathcal{CN}(\underline{\mu}, 2\sigma^2 I)$, find $\Pr\big(\|\underline{Z} - \underline{\mu}\| \geq \|\underline{Z} - \underline{c}\|\big)$.

– Sol. Note that

$$
\begin{aligned}
&\Pr\big(\|\underline{Z} - \underline{\mu}\| \geq \|\underline{Z} - \underline{c}\|\big) = \Pr\big(\|\underline{Z} - \underline{\mu}\|^2 \geq \|\underline{Z} - \underline{c}\|^2\big) \\
&= \Pr\big(\|\underline{Z}\|^2 - 2\,\mathrm{Re}\{\underline{\mu}^H \underline{Z}\} + \|\underline{\mu}\|^2 \geq \|\underline{Z}\|^2 - 2\,\mathrm{Re}\{\underline{c}^H \underline{Z}\} + \|\underline{c}\|^2\big) \\
&= \Pr\left(\mathrm{Re}\left\{\left(\underline{c} - \underline{\mu}\right)^H \underline{Z}\right\} \geq \frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2}\right) \\
&= \Pr\left(\mathrm{Re}\{W\} \geq \frac{\|\underline{c}\|^2 - \|\underline{\mu}\|^2}{2}\right).
\end{aligned}
$$

Since the complex Gaussian random variable $W \triangleq \left(\underline{c} - \underline{\mu}\right)^H \underline{Z}$ has

$$
\begin{aligned}
\mathbb{E}[W] &= \left(\underline{c} - \underline{\mu}\right)^H \underline{\mu} \\
\mathrm{Cov}[W] &= \left(\underline{c} - \underline{\mu}\right)^H \left(2\sigma^2 I\right)\left(\underline{c} - \underline{\mu}\right),
\end{aligned}
$$

we have

$$
\begin{aligned}
&\Pr\bigl(\|\underline{Z}-\underline{\mu}\| \geq \|\underline{Z}-\underline{c}\|\bigr) \\
&= Q\left(\frac{\frac{\|\underline{c}\|^2-\|\underline{\mu}\|^2}{2} - \left[\mathrm{Re}\{\underline{c}^H\underline{\mu}\} - \|\underline{\mu}\|^2\right]}{\sigma\|\underline{c}-\underline{\mu}\|}\right) \\
&= Q\left(\frac{\frac{\|\underline{c}-\underline{\mu}\|^2}{2}}{\sigma\|\underline{c}-\underline{\mu}\|}\right) = Q\left(\frac{\|\underline{c}-\underline{\mu}\|}{2\sigma}\right).
\end{aligned}
$$

Fig. (Geometrical interpretation)

– Example 2. When $\underline{Z} \sim \mathcal{CN}(\underline{\mu}, C) \in \mathbb{C}^N$, find $\Pr\left(\|\underline{Z} - \underline{\mu}\| \geq \|\underline{Z} - \underline{p}\|\right)$.

– Sol. Note that

$$\Pr\left(\|\underline{Z} - \underline{\mu}\| \geq \|\underline{Z} - \underline{p}\|\right) = \Pr\left(\|\underline{Z} - \underline{\mu}\|^2 \geq \|\underline{Z} - \underline{p}\|^2\right)$$

$$= \Pr\left( \mathrm{Re}\left\{\left(\underline{p} - \underline{\mu}\right)^H \underline{Z}\right\} \geq \frac{\|\underline{p}\|^2 - \|\underline{\mu}\|^2}{2} \right).$$

Let $W \triangleq \left(\underline{p} - \underline{\mu}\right)^H \underline{Z}$. Then,

$$\mathbb{E}[W] = (\underline{p} - \underline{\mu})^H \underline{\mu}$$

$$\mathrm{Cov}[W] = (\underline{p} - \underline{\mu})^H C (\underline{p} - \underline{\mu}).$$

Thus, we have

$$\Pr\left(\|\underline{Z} - \underline{\mu}\| \geq \|\underline{Z} - \underline{p}\|\right)$$

$$= \Pr\left( \mathrm{Re}\{W\} \geq \frac{\|\underline{p}\|^2 - \|\underline{\mu}\|^2}{2} \right)$$

$$= Q\left( \frac{\frac{\|\underline{p}\|^2 - \|\underline{\mu}\|^2}{2} - \mathrm{Re}\{(\underline{p} - \underline{\mu})^H \underline{\mu}\}}{\sqrt{\frac{1}{2}(\underline{p} - \underline{\mu})^H C (\underline{p} - \underline{\mu})}} \right) = Q\left( \frac{\|\underline{p} - \underline{\mu}\|^2}{\sqrt{2(\underline{p} - \underline{\mu})^H C (\underline{p} - \underline{\mu})}} \right).$$