Continuous-Time Random Processes: Time-Domain Characterization

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- Review. The received signal contains a noise $N(t)$ often modeled by **a continuous-time (CT) random process** called an additive white **Gaussian** noise (AWGN). To understand it, we started to review a pre-requisite of this course, *Probability, Random Variables, and Random Processes*.
  - statistical independence of multiple events, and multiple random variables
  - correlation, covariance, and correlation coefficient of two real-valued random variables, and two complex-valued random variables.
  - real-valued and complex-valued Gaussian random vectors

- Preview. We will study the notion of **a CT random process**.
  - How to characterize a CT random process in TD
  - Independence of CT random processes
  - Nth-order and strict-sense stationarity of a CT random process

- How to mathematically model an electrical signal
  - scalar x
  - vector $\underline{x}$
  - matrix
  - sequence of
    - scalars $x[n]$
    - vectors $\underline{x}[n]$
    - matrices
  - waveform (continuous-time function) of
    - scalar $x(t)$
    - vector $\underline{x}(t)$
    - matrix
  - field
    - scalar-valued $V(x, y, z)$
    - vector-valued $\underline{E}(x, y, z), \underline{B}(x, y, z)$
  - polynomial
  - graph, etc.

- How to **mathematically model** an electrical signal with uncertainty
  - unknown but not random
  - unknown and random

- How to **probabilistically model** an electrical signal with uncertainty
  - random variable
  - random vector
  - random matrix
  - DT random process
  - CT random process
  - random field, etc.

- Random Process Concept
  - Q. What is a random variable?
  - A. Given a probability space $(\Omega, \mathcal{A}, P)$, a random variable denoted by $X(\omega)$ or, simply, X is defined as a function from a sample space to an extended real line.
    - Fig.

  - Definition. Given a probability space $(\Omega, \mathcal{A}, P)$, a CT random process is defined as a mapping rule from a sample space to a set of CT signals.
  - Notation. A CT random process is denoted by $X(\omega, t)$ or, simply, by $X(t)$.

* Example. $\Pr(\{0\}) = 1/3, \Pr(\{1\}) = 2/3$.
  $X(0,t) = \cos(2\pi t), \forall t$
  $X(1,t) = \sin(2\pi t), \forall t$
* Fig.

– Definition. Each element of the set of CT signals is called **a sample path** or **an ensemble**.

- 2 Different Views to a CT Random Process
  - $X(\omega, t)$
    - Fig. Example (Graphs of $X(0,t)$ and $X(1,t)$)

    - Given a fixed ω_0, $X(\omega_0, t)$ is a deterministic CT signal.
      - The first view point is very useful in understanding a CT random process.
    - Given a fixed t_0, $X(\omega, t_0)$ is a random variable.
      - According to the second view point, **a CT random variable is a collection of uncountably infinite number of jointly distributed random variables**.
      - The second view point is very useful in characterizing a CT random process.

- How to Fully Characterize a CT Random Process
  - Q. How do we fully characterize a random vector $\underline{X}$, a collection of finite number of jointly distributed random variables?
  - A.
    - Method 1. Provide a probability space $(\Omega, \mathcal{A}, P)$ and a mapping rule $\underline{X}(\omega)$.
    - Method 2. Provide a joint CDF/PDF/CF of the finite number of random variables.
      - $X \rightarrow$ Provide the cdf $F_X(x)$.
      - $\underline{X} = \begin{bmatrix} X_1 \\ Y_2 \end{bmatrix} \rightarrow$ Provide the joint cdf $F_{X_1,X_2}(x_1, x_2)$.
      - $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} \rightarrow$ Provide the joint cdf $F_{X_1,X_2,X_3}(x_1, x_2, x_3)$.
      - $\underline{X} = \begin{bmatrix} X_1 \\ X_2 \\ \vdots \\ X_{10^6} \end{bmatrix} \rightarrow$ Provide the joint cdf $F_{X_1,X_2,\cdots,X_{10^6}}(x_1, x_2, \cdots, x_{10^6})$.

– Q. How do we fully characterize a CT random process $X(t)$, a collection of **infinite** number of jointly distributed random variables?

– A.

* Method 1. Provide a probability space $(\Omega, \mathcal{A}, P)$ and a mapping rule $X(\omega, t)$.
* Method 2. Provide all possible joint CDF/PDF/CF of the infinite number of random variables.
  · Theorem. (**Kolmogorov's Extension Theorem**) A CT random process is fully characterized by providing a joint CDF $F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1, x_2, \cdots, x_N)$ for every N-tuple $(t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$ for every $N \in \mathbb{N}$.

    (1) $F_{X(t)}(x)$ for every $t \in \mathbb{R}$,
    (2) $F_{X(t_1),X(t_2)}(x_1, x_2)$ for every pair $(t_1, t_2) \in \mathbb{R}^2$,
    (3) $F_{X(t_1),X(t_2),X(t_3)}(x_1, x_2, x_3)$ for every triplet $(t_1, t_2, t_3) \in \mathbb{R}^3$,
    (..)

  · Proof. Omitted.

– Q. How do we fully characterize two or more CT random processes?

– Q. How do we define statistical independence of two or more CT random processes?

- Nth-Order and Strict-Sense Stationarity of a CT Random Process
  - Definition. A CT random process $X(t)$ is **1st-order stationary** if $F_{X(t_1)}(x) = F_{X(t_2)}(x), \forall x, \forall t_1 \neq t_2$.
    - Fig.

  - Definition. A CT random process $X(t)$ is **2nd-order stationary** if $F_{X(t_1),X(t_2)}(x_1, x_2) = F_{X(t_1+\Delta),X(t_2+\Delta)}(x_1, x_2), \forall (x_1, x_2), \forall (t_1, t_2), \forall \Delta$.
    - Property
      - If $X(t)$ is 2nd-order stationary, then it is 1st-order stationary. The converse is not true in general.
      - Fig.

– Definition. A CT random process $X(t)$ is N**th-order stationary** if $F_{X(t_1),X(t_2),\cdots,X(t_N)}(x_1, x_2, \cdots, x_N) = F_{X(t_1+\Delta),X(t_2+\Delta),\cdots,X(t_N+\Delta)}$ $(x_1, x_2, \cdots, x_N), \forall(t_1, t_2, \cdots, t_N) \in \mathbb{R}^N, \forall\Delta$.

– Definition. A CT random process $X(t)$ is **strictly stationary** if it is Nth-order stationary for every $N \in \mathbb{N}$.

∗ Fig. Venn Diagram