CT Random Processes: Expectation, Correlation, and Wide-Sense Stationarity

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- Review. The received signal contains a noise $N(t)$ often modeled by **a continuous-time (CT) random process** called an additive white Gaussian noise (AWGN). To understand it, we started to review a pre-requisite of this course, *Probability, Random Variables, and Random Processes.*
  - notion of a CT random process
  - How to characterize a CT random process in TD
  - Independence of CT random processes
  - Nth-order and strict-sense stationarity of a CT random process

- Preview.
  - expectation of a CT random process
  - correlation functions of a CT random process
  - a wide-sense stationary (WSS) CT random process
  - joint WSS CT random processes

- Expectation/Mean of a CT Random Process
  - Q. What is the expectation of a CT random process $\mathbb{E}\{X(t)\}$?
    - Q. What is the expectation of a random variable?
    - Q. What is the expectation of a random vector?
    - Hint: What is $X(t_0)$?
    - A. The expectation is defined as

    - Properties
      - In general, $\mathbb{E}\{X(t)\}$ is a deterministic CT signal, a function of t.
      - Thus, it is called the mean path, mean process, or mean function of $X(t)$.
      - It is often denoted by $\overline{X}(t)$ or $\mu_X(t)$.
      - There are some cases where $\mathbb{E}\{X(t)\} = \text{constant}, \forall t$.

– Q. Show that, if $X(t)$ is 1st-order stationary, there exists a constant μ_X such that $\mathbb{E}\{X(t)\} = \mu_X, \forall t$. Also show that the converse is not true in general.

– A.

  * (Direct Part)

  * (Converse) A counter example

– If $X(t)$ is Nth-order stationary for some $N \in \mathbb{N}$, then the mean function is a constant. Why?

- Auto-Correlation and Auto-Covariance Functions of a CT Random Process
  - Definition. Given a **real-valued** CT random process $X(t)$, the **auto-correlation function of** $X(t)$ is denoted by $R_{XX}(t_1, t_2)$ and defined by

$$R_{XX}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)X(t_2)\}.$$

    * A **complex-valued** CT random process has two auto-correlation functions:
      - the auto-correlation function: $R_{XX}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)X(t_2)^*\}$
      - the **complementary/pseudo auto-correlation function**: $\tilde{R}_{XX}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)X(t_2)\}$
    * Properties
      - In general, the autocorrelation function is a two-variable function.
      - There are some cases where $R_{XX}(t_1, t_2)$ is a single-variable function.

– Q. Show that, if $X(t)$ is 2nd-order stationary, $R_{XX}(t_1, t_2)$ is a single-variable function.

– A.

  * $R_{XX}(t_1, t_2) = R_{XX}(t_1 + \Delta, t_2 + \Delta), \forall \Delta$
  * There exists a function $R_{XX}(\tau)$ such that $R_{XX}(\tau) = R_{XX}(t + \tau, t), \forall t, \forall \tau$
  * The auto-correlation is a function only of a time difference $\tau = t_1 - t_2$.
  * The converse is not true in general.

– If $X(t)$ is Nth-order stationary with $N \in \mathbb{N}\backslash\{1\}$, then the auto-correlation is a function only of a time difference. Why?

– Definition. Given a **real-valued** CT random process $X(t)$, the **auto-covariance function of** $X(t)$ is denoted by $C_{XX}(t_1, t_2)$ and defined by

$$C_{XX}(t_1, t_2) \triangleq \mathbb{E}[\{X(t_1) - \overline{X}(t_1)\}\{X(t_2) - \overline{X}(t_2)\}].$$

* If $Y(t) = X(t) - \overline{X}(t)$, then $C_{XX}(t_1, t_2) = R_{YY}(t_1, t_2)$.
* A **complex-valued** CT random process has two auto-covariance functions:
  - the auto-covariance function: $C_{XX}(t_1, t_2) \triangleq R_{YY}(t_1, t_2)$
  - the **complementary/pseudo auto-covariance function**: $\tilde{C}_{XX}(t_1, t_2) \triangleq \tilde{R}_{YY}(t_1, t_2)$
* Properties
  - Since an auto-covariance function is an auto-correlation function, the properties are inherited.
  - In general, an auto-covariance function is a two-variable function.
  - There are some cases where $C_{XX}(t_1, t_2)$ is a single-variable function.

- Definition. A real-valued CT random process $X(t)$ is **covariance stationary** if $C_{XX}(t_1, t_2)$ is a function only of the time difference, i.e., there exists $C_{XX}(\tau) = C_{XX}(t + \tau, t), \forall t, \forall \tau$.
- Definition. A complex-valued CT random process $X(t)$ is **proper** if $\tilde{C}_{XX}(t_1, t_2)$ vanishes, i.e., $\tilde{C}_{XX}(t_1, t_2) = 0, \forall (t_1, t_2)$.

• Wide-Sense Stationarity of a CT Random Process

- Definition. A real-valued CT random process $X(t)$ is **wide-sense stationary (WSS)** if

(a) $\mathbb{E}\{X(t)\}$ is constant, and

(b) $\mathbb{E}\{X(t + \tau)X(t)\}$ is a function only of τ.

∗ Property

· If $X(t)$ is 2nd-order stationary, then it is WSS. The converse is not true in general.

- If $X(t)$ is WSS, then it is covariance stationary because

$$C_{XX}(t+\tau,t) = ... = R_{XX}(\tau) - \mu_X^2.$$

  However, the converse is not true in general because

$$R_{XX}(t+\tau,t) = ... = C_{XX}(\tau) + \mu_X(t+\tau)\mu_X(t).$$

- If $X(t)$ is a zero-mean WSS random process, then wide-sense stationarity is equivalent to covariance stationarity.

– Fig. Venn Diagram

- Joint Wide-Sense Stationarity of Two Real-Valued CT Random Processes
  - Definition. Given two **real-valued** CT random processes $X(t)$ and $Y(t)$, the **cross-correlation function between $X(t)$ and $Y(t)$** is denoted by $R_{XY}(t_1, t_2)$ and defined by
$$R_{XY}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)Y(t_2)\}.$$
    - For two **complex-valued** CT random processes, we have the cross-correlation function $R_{XY}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)Y(t_2)^*\}$ and the **complementary/pseudo cross-correlation function** $\tilde{R}_{XY}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)Y(t_2)\}$.
  - Definition. Given two **real-valued** CT random processes $X(t)$ and $Y(t)$, the **cross-covariance function between $X(t)$ and $Y(t)$** is denoted by $C_{XY}(t_1, t_2)$ and defined by
$$C_{XY}(t_1, t_2) \triangleq \mathbb{E}[\{X(t_1) - \overline{X}(t_1)\}\{Y(t_2) - \overline{Y}(t_2)\}].$$
    - For two **complex-valued** CT random processes, we have the cross-covariance function $C_{XY}(t_1, t_2)$ and the **complementary/pseudo cross-covariance function** $\tilde{C}_{XY}(t_1, t_2)$.

– Definition. Two **real-valued** CT random processes $X(t)$ and $Y(t)$ are **jointly wide-sense stationary** if

(a) $X(t)$ is WSS,

(b) $Y(t)$ is WSS, and

(c) $R_{XY}(t+\tau, t)$ is a function only of τ.

* Properties
  - mean functions
  - auto-correlation functions