CT Random Processes: Frequency-Domain Char-acterization

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- Review. The received signal contains a noise $N(t)$ often modeled by **a continuous-time (CT) random process** called an additive white Gaussian noise (AWGN). To understand it, we started to review a pre-requisite of this course, *Probability, Random Variables, and Random Processes*.
  - notion of a CT random process
  - How to characterize a CT random process in TD
  - Independence of CT random processes
  - correlation functions of a CT random process
  - wide-sense stationary (WSS) CT random processes

- Preview.
  - Energy spectral density (ESD) and power spectral density (PSD) of deterministic CT signals
  - PSD of a CT random process
  - cross PSD of two CT random processes

- Energy Spectral Density (ESD) of a Finite-Energy Deterministic CT Signal
  - Definition. **The energy of a deterministic CT signal** $x(t)$ is defined as
$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t.$$
    * The energy is a non-negative real number.
  - Definition. If $0 < E_{xx} < \infty$, then $x(t)$ is called **a finite-energy signal** or an energy signal.
  - Definition. When $x(t) \overset{\mathcal{F}}{\leftrightarrow} X(f)$, **the energy spectral density (ESD)** of an energy signal $x(t)$ is defined as
$$|X(f)|^2.$$
    * The ESD is a non-negative real function of f. Moreover, it is symmetric if $x(t)$ is real.

* The integration of the ESD leads to E_{xx} because

$$\int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t = \int_{-\infty}^{\infty} |X(f)|^2 \mathrm{d}f.$$

* Fig. Bandpass filtering of $x(t)$ at f_0 with bandwidth ϵ

$$\begin{aligned}\int_{-\infty}^{\infty} |y(t)|^2 \, \mathrm{d}t &= \int_{-\infty}^{\infty} |Y(f)|^2 \, \mathrm{d}f \\ &= \int_{-\infty}^{\infty} |X(f)|^2 |H(f)|^2 \, \mathrm{d}f \\ &\cong 2\epsilon |X(f_0)|^2\end{aligned}$$

by Parseval's theorem.

* In general, the ESD does NOT fully characterize the signal. It partly characterizes the signal in the FD because $x(t)$ cannot be recovered from $|X(f)|^2$.

- Instantaneous Power vs. Average Power
  - Energy revisited:

$$E_{xx} \triangleq \int_{-\infty}^{\infty} |x(t)|^2 \mathrm{d}t.$$

    * power = energy per unit time
    * energy = power times time
    * Thus, $|x(t)|^2$ is called **the instantaneous power** of $x(t)$.
    * The instantaneous power is a non-negative real number.

  - Definition. **The average power of a deterministic CT signal** $x(t)$ is defined as

$$P_{xx} \triangleq \mathbb{A}\{|x(t)|^2\} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} |x(t)|^2 \mathrm{d}t.$$

    where $\mathbb{A}\{\cdot\}$ denotes **the time-average**.
    * The average power is a non-negative real number.

  - Definition. If $0 < P_{xx} < \infty$, then $x(t)$ is called **a finite-power signal** or a power signal.

– Note that

$$0 < P_{xx} < \infty \implies E_{xx} = \infty$$
$$0 < E_{xx} < \infty \implies P_{xx} = 0$$

- Power Spectral Density (PSD) of a Finite-Power Deterministic CT Signal

  – Definition. When

$$x_\Delta(t) = \begin{cases} x(t), & -\Delta < t < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

and $x_\Delta(t) \overset{\mathcal{F}}{\leftrightarrow} X_\Delta(f)$, **the power spectral density (PSD)** of a power signal $x(t)$ is defined as

$$S_{xx}(f) \triangleq \lim_{\Delta \to \infty} \frac{|X_\Delta(f)|^2}{2\Delta}.$$

  * The PSD is a non-negative real function of f. If $x(t)$ is real, then it is symmetrical.

* Note that the PSD is the limit of the ESD $|X_\Delta(f)|^2$ divided by time 2Δ.

* The integration of the PSD leads to P_{xx} because

$$\begin{aligned} P_{xx} &= \lim_{\Delta\to\infty} \int_{-\Delta}^{\Delta} \frac{|x(t)|^2}{2\Delta} \mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{|x_\Delta(t)|^2}{2\Delta} \mathrm{d}t \\ &= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{|X_\Delta(f)|^2}{2\Delta} \mathrm{d}f \\ &= \int_{-\infty}^{\infty} \lim_{\Delta\to\infty} \frac{|X_\Delta(f)|^2}{2\Delta} \mathrm{d}f \\ &= \int_{-\infty}^{\infty} S_{xx}(f) \mathrm{d}f \quad \text{(if the limit exists)} \end{aligned}$$

* In general, the PSD does NOT fully characterize the signal. It partly characterizes the signal in the FD because $x(t)$ cannot be recovered from $S_{xx}(f)$.

- Cross-Power Spectral Density (xPSD) of Two Finite-Power Deterministic CT Signals
  - Q. Given two finite-power deterministic CT power signals $x(t)$ and $y(t)$, what is the PSD of $w(t) = x(t) + y(t)$?
  - A.
    * In general, $P_{ww} \neq P_{xx} + P_{yy}$.
      · Definition. The cross power between power signals $x(t)$ and $y(t)$ is defined as

$$P_{xy} \triangleq \mathbb{A}\{x(t)y^*(t)\}.$$

      · In general, the cross power is complex-valued. If $x(t)$ and $y(t)$ are real, then it is a real number that is positive, negative, or zero.

      · Thus, $P_{ww} = P_{xx} + P_{xy} + P_{yx} + P_{yy} = P_{xx} + 2\,\mathrm{Re}\{P_{xy}\} + P_{yy}$.

* Similarly, $S_{ww}(f) \neq S_{xx}(f) + S_{yy}(f)$ in general.
  - Definition. **The cross-power spectral density (xPSD)** between power signals $x(t)$ and $y(t)$ is defined as

$$S_{xy}(f) \triangleq \lim_{\Delta \to \infty} \frac{X_\Delta(f) Y_\Delta^*(f)}{2\Delta}.$$

  - In general, the xPSD is a complex-valued function. If $x(t)$ and $y(t)$ are real, then it is a conjugate symmetrical function.

* Thus, $S_{ww}(f) = S_{xx}(f) + S_{xy}(f) + S_{yx}(f) + S_{yy}(f) = S_{xx}(f) + 2\,\mathrm{Re}\{S_{xy}(f)\} + S_{yy}(f)$.

- Time Average of a CT Random Process
  - Q. Given a CT random process $X(t)$, what is **the time average** $\mathbb{A}\{X(t)\}$ of it?
  - A. It is a random variable because

$$\mathbb{A}\{X(t)\} \triangleq \lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} X(t)\mathrm{d}t$$

    is a scalar with probabilistic description.

    * cf. **The ensemble average** or statistical average $\mathbb{E}\{X(t)\}$ is a deterministic CT signal.

    * Example. Given $\Pr(X(t) = \cos(2\pi t)) = \Pr(X(t) = 0) = \frac{1}{2}$, find $\mathbb{E}\{|X(t)|^2\}$ and $\mathbb{A}\{|X(t)|^2\}$.

$$\mathbb{E}\{|X(t)|^2\} = \frac{1}{2}\cos^2(2\pi t)$$

      and

$$\Pr(\mathbb{A}\{|X(t)|^2\} = 1/2) = \Pr(\mathbb{A}\{|X(t)|^2\} = 0) = 1/2.$$

- PSD of a CT Random Process
  - Definition. **The averaged power of a CT random process** $X(t)$ is defined as

$$P_{XX} \triangleq \mathbb{A}[\mathbb{E}\{|X(t)|^2\}].$$

    * This definition is backward compatible with that for a deterministic CT signal $x(t)$.
    * It is the time average of the ensemble average of $|X(t)|^2$.
    * $P_{XX} = \mathbb{E}[\mathbb{A}\{|X(t)|^2\}]$.
    * Thus, it is a deterministic non-negative real number.

    * Example.

– Definition. When

$$X_\Delta(t) = \begin{cases} X(t), & -\Delta < t < \Delta \\ 0, & \text{elsewhere} \end{cases}$$

and $X_\Delta(f) \overset{\mathcal{F}}{\leftrightarrow} X_\Delta(t)$, **the PSD** of a CT random process $X(t)$ is defined as

$$S_{XX}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\left[|X_\Delta(f)|^2\right]}{2\Delta}$$

* This definition is backward compatible with that for a deterministic CT signal $x(t)$.
* The PSD is a non-negative real function of f. If $X(t)$ is real, then it is symmetrical.
* Note that the PSD is the limit of $\mathbb{E}\{|X_\Delta(f)|^2\}$ divided by time 2Δ.

* The integration of the PSD leads to P_{XX} because

$$\begin{aligned}
P_{XX} &\triangleq \mathbb{A}\big[\mathbb{E}\big[|X(t)|^2\big]\big] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} \mathbb{E}\big[|X(t)|^2\big]\ \mathrm{d}t \\
&= \lim_{\Delta\to\infty} \int_{-\infty}^{\infty} \frac{\mathbb{E}\big[|X_\Delta(t)|^2\big]}{2\Delta}\,\mathrm{d}t \\
&= \lim_{\Delta\to\infty} \mathbb{E}\left[\int_{-\infty}^{\infty} \frac{|X_\Delta(f)|^2}{2\Delta}\,\mathrm{d}f\right] \\
&= \int_{-\infty}^{\infty} \lim_{\Delta\to\infty} \frac{\mathbb{E}\big[|X_\Delta(f)|^2\big]}{2\Delta}\,\mathrm{d}f \\
&= \int_{-\infty}^{\infty} S_{XX}(f)\,\mathrm{d}f
\end{aligned}$$

* In general, the PSD does NOT fully characterize the CT random process. It partly characterizes the signal in the FD.

- xPSD of Two CT Random Processes
  - Q. Given two CT random processes $X(t)$ and $Y(t)$, what is the PSD of $W(t) = X(t) + Y(t)$?
  - A.
    * In general, $P_{WW} \neq P_{XX} + P_{YY}$.
      · Definition. The cross power between CT random processes $X(t)$ and $Y(t)$ is defined as

$$P_{XY} \triangleq \mathbb{A}[\mathbb{E}\{X(t)Y^*(t)\}].$$

      · This definition is backward compatible with that for deterministic CT signals $x(t)$ and $y(t)$.
      · In general, the cross power is complex-valued. If $X(t)$ and $Y(t)$ are real, then it is a real number that is positive, negative, or zero.
      · Thus, $P_{WW} = P_{XX}+P_{XY}+P_{YX}+P_{YY} = P_{XX}+2\,\mathrm{Re}\{P_{XY}\}+P_{YY}$.

    * Similarly, $S_{WW}(f) \neq S_{XX}(f) + S_{YY}(f)$ in general.

· Definition. **The xPSD** between two CT random processes $X(t)$ and $Y(t)$ is defined as

$$S_{XY}(f) \triangleq \lim_{\Delta\to\infty} \frac{\mathbb{E}\{X_\Delta(f)Y_\Delta^*(f)\}}{2\Delta}.$$

· In general, the xPSD is a complex-valued function. If $X(t)$ and $Y(t)$ are real, then it is a conjugate symmetrical function.

* Thus, $S_{WW}(f) = S_{XX}(f) + S_{XY}(f) + S_{YX}(f) + S_{YY}(f) = S_{XX}(f) + 2\,\mathrm{Re}\{S_{XY}(f)\} + S_{YY}(f)$.

- Q. Why is the PSD so important in wireless communications, though it partially characterizes a random process?
- A.
  - The signal transmitted by a Tx is well modeled by a CT real-valued bandpass random process.
  - Regulatory organizations impose restrictions on the peak power, the average power, the PSD, etc. of the transmitted signal.
  - spectral mask
  - Thus, it is crucial to design standards and systems that do not violate such restrictions.