CT Random Processes: PSD and xPSD

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- Review. The transmitted signal is well modeled by **a real-valued continuous-time (CT) bandpass random process**. Moreover, the received signal contains a noise $N(t)$ often modeled by **a real-valued CT bandpass random process** called an additive white Gaussian noise (AWGN).
  - TD full/partial characterization of CT random processes
    - Kolmogorov's extension theorem
    - auto-correlation and auto-covariance functions
    - cross-correlation and cross-covariance functions
    - wide-sense stationarity and joint wide-sense stationarity
  - FD partial characterization of CT random processes
    - PSD of a CT random process
    - xPSD of two CT random processes

- Preview.
  - Relation between the auto-correlation function and the PSD
  - Relation between the cross-correlation function and the xPSD

- Q. Recall that the PSD of a CT random process $X(t)$ is defined as

$$S_{XX}(f) \triangleq \lim_{\Delta \to \infty} \frac{\mathbb{E}\left[|X_\Delta(f)|^2\right]}{2\Delta}.$$

Is there any alternative form with which we can easily compute the PSD?

  - Notice that the PSD is related only to the second-order moment of $X(t)$.
  - The 2nd-order TD characterization is done by the auto-correlation function defined as

$$R_{XX}(t+\tau, t) \triangleq \mathbb{E}\{X(t+\tau)X^*(t)\}.$$

  - The question is to find the relation between $R_{XX}(t+\tau, t)$ and $S_{XX}(f)$.

- A. The PSD can be found from the auto-correlation function as

$$S_{XX}(f) = \mathcal{F}\{\mathbb{A}[R_{XX}(t+\tau, t)]\} \,.$$

  - $S_{XX}(f) = \mathcal{F}\{\mathbb{A}[\mathbb{E}\{X(t+\tau)X^*(t)\}]\}$
  - The PSD is NOT affected by the complementary/pseudo auto-correlation function $\tilde{R}_{XX}(t+\tau, t)$.
  - $\mathbb{A}[R_{XX}(t+\tau, t)]$ is a function only of τ and often denoted by $\mathcal{R}_{XX}(\tau)$.
  - $\mathcal{R}_{XX}(\tau) \xleftrightarrow{\mathcal{F}} S_{XX}(f)$
- (Wiener-Khinchin Theorem) If $X(t)$ is WSS, then

$$R_{XX}(\tau) \xleftrightarrow{\mathcal{F}} S_{XX}(f),$$

i.e.,

$$\begin{cases} S_{XX}(f) = \mathcal{F}\{R_{XX}(\tau)\} = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\ R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f) e^{\mathrm{j}2\pi f\tau}\,\mathrm{d}f. \end{cases}$$

– Example. Find the PSD of

$$X(t) = \cos(2\pi f_0 t) = \frac{e^{\mathrm{j}2\pi f_0 t} + e^{-\mathrm{j}2\pi f_0 t}}{2}, \forall t$$
$$\xleftrightarrow{\mathcal{F}} \quad \frac{1}{2}\delta(f - f_0) + \frac{1}{2}\delta(f + f_0).$$

* $X(t)$ is deterministic, so that no statistical average is required.
* Since $R_{XX}(t+\tau, t)$ is given by

$$R_{XX}(t+\tau, t) = \cos 2\pi f_0(t+\tau) \cos 2\pi f_0(t)$$
$$= \frac{1}{2}\left\{\cos(2\pi f_0(2t+\tau)) + \cos(2\pi f_0 \tau)\right\},$$

$\mathcal{R}_{XX}(\tau)$ is obtained as

$$\mathbb{A}[R_{XX}(t+\tau, t)] = \frac{1}{2}\cos(2\pi f_0 \tau).$$

Thus,

$$S_{XX}(f) = \frac{1}{4}\delta(f - f_0) + \frac{1}{4}\delta(f + f_0).$$

– (Generalization) The PSD of $X(t) = \sum_{i \in I} a_i e^{j2\pi f_i t}$ is given by

$$S_{XX}(f) = \sum_{i \in I} |a_i|^2 \delta(f - f_i).$$

∗ Fig. Graphs of $X(f)$ and $S_{XX}(f)$

∗ The PSD is NOT a full characterization of the random process. Later, we will see that it is the case when $X(t)$ is a real-valued WSS Gaussian random process.

- Q. Recall that the xPSD of two CT random processes $X(t)$ and $Y(t)$ is defined as

$$S_{XY}(f) \triangleq \lim_{\Delta \to \infty} \frac{X_\Delta(f) Y_\Delta^*(f)}{2\Delta}.$$

Is there any alternative form with which we can easily compute the xPSD?

- A. The xPSD can be found from the cross-correlation function as

$$S_{XY}(f) = \mathcal{F}\{\mathbb{A}[R_{XY}(t+\tau, t)]\}\,.$$

  - $S_{XY}(f) = \mathcal{F}\{\mathbb{A}[\mathbb{E}\{X(t+\tau)Y^*(t)\}]\}$
  - The PSD is NOT affected by the complementary/pseudo cross-correlation function $\tilde{R}_{XY}(t+\tau, t)$.
  - $\mathbb{A}[R_{XY}(t+\tau, t)]$ is a function only of τ and often denoted by $\mathcal{R}_{XY}(\tau)$.
  - $\mathcal{R}_{XY}(\tau) \xleftrightarrow{\mathcal{F}} S_{XY}(f)$
  - If $X(t)$ and $Y(t)$ are jointly WSS, then

$$R_{XY}(\tau) \xleftrightarrow{\mathcal{F}} S_{XY}(f),$$

- Proof.

$$
\begin{aligned}
S_{XX}(f) &= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[|X_\Delta(f)|^2\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[X_\Delta(f)\,(X_\Delta(f))^*\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\mathbb{E}\left[\left(\int_{-\infty}^{\infty} X_\Delta(t_1)e^{-\mathrm{j}2\pi f t_1}\,\mathrm{d}t_1\right)\left(\int_{-\infty}^{\infty} X_\Delta(t_2)e^{-\mathrm{j}2\pi f t_2}\,\mathrm{d}t_2\right)^*\right] \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\int_{-\Delta}^{\Delta} \mathbb{E}[X(t_1)X^*(t_2)]\, e^{-\mathrm{j}2\pi f(t_1-t_2)}\,\mathrm{d}t_1\,\mathrm{d}t_2 \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta}^{\Delta}\int_{-\Delta-t}^{\Delta-t} \mathbb{E}[X(t+\tau)X^*(t)]\, e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau\,\mathrm{d}t \\
&= \lim_{\Delta\to\infty} \frac{1}{2\Delta}\int_{-\Delta-t}^{\Delta-t}\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}t\,\mathrm{d}\tau \\
&= \lim_{\Delta\to\infty} \int_{-\Delta-t}^{\Delta-t}\frac{1}{2\Delta}\left(\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)\,\mathrm{d}t\right) e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\
&= \int_{-\infty}^{\infty}\left(\lim_{\Delta\to\infty}\frac{1}{2\Delta}\int_{-\Delta}^{\Delta} R_{XX}(t+\tau,t)\,\mathrm{d}t\right) e^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\
&= \mathcal{F}\{\mathbb{A}[R_{XX}(t+\tau,t)]\}
\end{aligned}
$$

- Properties of $S_{XX}(f)$ and $S_{XY}(f)$
  - Use either the definition or the theorem.
  - (real, non-negative PSD) $S_{XX}(f) \geq 0, \forall f$.
  - $\dot{X}(t) \triangleq \frac{\mathrm{d}}{\mathrm{d}t} X(t)$
    $S_{\dot{X}\dot{X}}(f) = (2\pi f)^2 S_{XX}(f)$
  - (conjugate pair) $S_{XY}(f) = S_{YX}^{*}(f), \forall f$.
    - $\mathrm{Re}\{S_{XY}(f)\} = \mathrm{Re}\{S_{YX}(f)\}$ and $\mathrm{Im}\{S_{XY}(f)\} = -\mathrm{Im}\{S_{YX}(f)\}$.
  - If $X(t)$ and $Y(t)$ are orthogonal, i.e., $R_{XY}(t+\tau, t) = 0, \forall t, \forall \tau$, then $S_{XY}(f) = 0, \forall f$. However, the converse is not true in general.

- If $X(t)$ and $Y(t)$ are jointly WSS, then
  * $R_{XX}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XX}(f)$
  * $R_{YY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{YY}(f)$
  * $R_{XY}(\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{XY}(f)$
  * $R_{YX}(\tau) = R^*_{XY}(-\tau) \overset{\mathcal{F}}{\longleftrightarrow} S_{YX}(f) = S^*_{XY}(f)$
  * $P_{XX} = R_{XX}(0), P_{YY} = R_{YY}(0), P_{XY} = R_{XY}(0) = R^*_{YX}(0) = P^*_{YX}$

- If $X(t)$ and $Y(t)$ are real-valued, then
  * (real, non-negative, symmetric PSD) $S_{XX}(f) = S_{XX}(-f), \forall f$

  * (conjugate symmetrical xPSD) $S_{XY}(f) = S^*_{XY}(-f), \forall f$