CT Random Processes: LTI Filtering of CT Random Process

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- Review. The transmitted signal is well modeled by **a real-valued continuous-time (CT) bandpass random process**. Moreover, the received signal contains a noise $N(t)$ often modeled by **a real-valued CT bandpass random process** called an additive white Gaussian noise (AWGN).
  - TD full/partial characterization of CT random processes
  - FD partial characterization of CT random processes
    - Auto-correlation function and PSD of a CT random process
    - Cross-correlation function and xPSD of two CT random processes

- Preview.
  - Rx front end
  - LTI filtering of a white noise
  - TD and FD characterization of the output of an LTI system to a CT random process input

- Receiver Front End
  - Q. What is common in the Rx front ends of a homodyne receiver and a heterodyne receiver?
  - A. Fig.
    - Antenna, Pre-Select Filter, Low Noise Amplifier
    - homodyne: channel select **filter**
    - heterodyne: image reject **filter**, mixer, and channel select filter
    - Note that the **additive white Gaussian noise** generated by the LNA **is filtered by a real-valued bandpass filter.**

- Definition. A real-valued CT random process $X(t)$ is called **white** if its PSD is given by

$$S_{XX}(f) = \frac{N_0}{2}, \forall f$$

for some N_0.

  - N_0 is called **the one-sided PSD of the white noise.**
  - $N_0/2$ is called **the two-sided PSD of the white noise.**
  - A white noise has an infinite power.
  - A white noise is an approximation for the case where the LTI system has a power transfer function inside the flat region of the PSD.
  - The output PSD and the power of an LTI system with impulse response $h(t)$ is given, respectively, by (The proof will be given soon.)

$$S_{YY} = \frac{N_0}{2}|H(f)|^2$$

and

$$P_{YY} = \frac{N_0}{2}\int_{-\infty}^{\infty}|h(t)|^2 dt = \frac{N_0}{2}\int_{-\infty}^{\infty}|H(f)|^2 df.$$

- Definition. A real-valued CT random process $X(t)$ is called **a band-limited white noise** if its PSD is given by

$$S_{XX}(f) = \begin{cases} \frac{N_0}{2}, \text{for } |f| \leq W \\ 0, \text{elsewhere} \end{cases}$$

or

$$S_{XX}(f) = \begin{cases} \frac{N_0}{2}, \text{for } |f - f_c| \leq W/2 \\ 0, \text{elsewhere} \end{cases}$$

for some N_0, W, and $f_c > W/2$.

  – A band-limited white noise is a filtered version of a white noise by an ideal LPF or an ideal BPF.

  – Figs.

- LTI filtering of a CT random process
  - Q. What is the output of a CT LTI system when the input is a CT random process?
  - A.
    - Fig.
    - For each sample path input, the output is a deterministic signal.
    - Since the output is a CT signal with uncertainty described probabilistically, **the output is a CT random process.**
  - The output $Y(t)$ as the sample path-by-sample path convolution with the impulse response $h(t)$ is denoted by

$$Y(t) = h(t) * X(t).$$

- TD Characterization of LTI filtered CT Random Process
  - Q. Given a CT random process $X(t)$ and an LTI system with impulse response $h(t)$, can we fully characterize the output $Y(t)$ in the TD?
  - A. Not in general.
    * Fig.

    * Even if the input $X(t)$ is fully characterized, the output $Y(t)$ is hard to characterize in general except some special cases such as $h(t) = \alpha\delta(t - t_0)$.

– Q. Given a CT random process $X(t)$ and an LTI system with impulse response $h(t)$, can we characterize the **2nd-order** properties of the output $Y(t)$ in the TD?

– A. Yes, we can.
If the auto-correlation function $R_{XX}(t_1, t_2)$ of $X(t)$ is given, then the cross-correlation functions are obtained as

$$R_{XY}(t_1, t_2) \triangleq \mathbb{E}\{X(t_1)Y^*(t_2)\}$$

$$= \int_{-\infty}^{\infty} R_{XX}(t_1, t_2 - \xi_2)h^*(\xi_2)d\xi_2$$
$$= R_{XX}(t_1, t_2) * h^*(t_2)$$

and

$$R_{YX}(t_1, t_2) \triangleq \mathbb{E}\{Y(t_1)X^*(t_2)\}$$

$$= \int_{-\infty}^{\infty} h(\xi_1)R_{XX}(t_1 - \xi_1, t_2)d\xi_1$$
$$= h(t_1) * R_{XX}(t_1, t_2),$$

and the auto-correlation function $R_{YY}(t_1, t_2)$ of $Y(t)$ is obtained as

$$R_{YY}(t_1, t_2) \triangleq \mathbb{E}\{Y(t_1)Y^*(t_2)\} = ...$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi_1) R_{XX}(t_1 - \xi_1, t_2 - \xi_2) h^*(\xi_2) d\xi_1 d\xi_2$$

$$= h(t_1) * R_{XX}(t_1, t_2) * h^*(t_2)$$

$$= h(t_1) * R_{XY}(t_1, t_2)$$

$$= R_{YX}(t_1, t_2) * h^*(t_2).$$

$*$ (Special Case) **If the input to an LTI system is WSS, then the input $X(t)$ and the output $Y(t)$ are jointly WSS** with

(a) $R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau)$,

(b) $R_{YX}(\tau) = R_{XX}(\tau) * h(\tau)$, and

(c) $R_{YY}(\tau) = h(\tau) * R_{XY}(\tau) = R_{YX}(\tau) * h^*(-\tau) = h(\tau) * R_{XX}(\tau) * h^*(-\tau)$.

because $R_{XY}(t_1, t_2)$ between $X(t)$ and $Y(t)$ is obtained as

$$\begin{aligned} R_{XY}(t_1, t_2) &= \int_{-\infty}^{\infty} R_{XX}(t_1, t_2 - \xi_2) h^*(\xi_2) d\xi_2 \\ &= \int_{-\infty}^{\infty} R_{XX}(t_1 - t_2 - \xi_2) h^*(-\xi_2) d\xi_2, \end{aligned}$$

and the auto-correlation function $R_{YY}(t_1, t_2)$ of $Y(t)$ is obtained as

$$\begin{aligned} R_{YY}(t_1, t_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi_1) R_{XX}(t_1 - \xi_1, t_2 - \xi_2) h^*(\xi_2) d\xi_1 d\xi_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi_1) R_{XX}(t_1 - t_2 - \xi_1 - \xi_2) h^*(-\xi_2) d\xi_1 d\xi_2. \end{aligned}$$

- FD Characterization of LTI filtered CT Random Process
  - Q. Given a CT random process $X(t)$ and an LTI system with impulse response $h(t)$, can we characterize the output $Y(t)$ in the FD?
  - A. The xPSDs and the output PSD are given by
    * $S_{XY}(f) = S_{XX}(f)H^*(f)$,
    * $S_{YX}(f) = H(f)S_{XX}(f)$, and
    * $S_{YY}(f) = S_{XX}(f)|H(f)|^2$.
    * Fig. block diagram
    * Note that **the xPSDs and the output PSD can be obtained from only**
      - **the PSD $S_{XX}(f)$ of the input random process and**
      - **the frequency response $H(f)$ of the LTI system.**
    * $|H(f)|^2$ is called **the power transfer function**.

$*$ (Proof) Recall that

$\cdot\ R_{XY}(t_1, t_2) = \int_{-\infty}^{\infty} R_{XX}(t_1, t_2 - \xi_2) h^*(\xi_2) d\xi_2$,

$\cdot\ R_{YX}(t_1, t_2) = \int_{-\infty}^{\infty} h(\xi_1) R_{XX}(t_1 - \xi_1, t_2) d\xi_1$, and

$\cdot\ R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi_1) R_{XX}(t_1 - \xi_1, t_2 - \xi_2) h^*(\xi_2) d\xi_1 d\xi_2$.

Thus,

$\cdot\ \mathcal{R}_{XY}(\tau) = \mathbb{A}\{R_{XY}(t + \tau, t)\} = ... = \mathcal{R}_{XX}(\tau) * h^*(-\tau)$,

$\cdot\ \mathcal{R}_{YX}(\tau) = \mathbb{A}\{R_{YX}(t + \tau, t)\} = h(\tau) * \mathcal{R}_{XX}(\tau)$, and

$\cdot\ \mathcal{R}_{YY}(\tau) = \mathbb{A}\{R_{YY}(t + \tau, t)\} = ... = h(\tau) * \mathcal{R}_{XX}(\tau) * h^*(-\tau)$.

Therefore, the conclusion follows.

- Cross Power between Input and Output, and Output Power
  - Q. Find the output power P_{YY} and the cross powers P_{XY} and P_{YX}, in terms of $R_{XX}(t_1, t_2)$ and $h(t)$.
  - A.
    * $P_{XY} = \int_{-\infty}^{\infty} S_{XX}(f)H^*(f)df = \mathcal{R}_{XX}(\tau) * h^*(-\tau)|_{\tau=0}$
    * $P_{YX} = \int_{-\infty}^{\infty} H(f)S_{XX}(f)df = h(\tau) * \mathcal{R}_{XX}(\tau)|_{\tau=0}$
    * $P_{YY} = \int_{-\infty}^{\infty} S_{XX}(f)|H(f)|^2 df = h(\tau) * \mathcal{R}_{XX}(\tau) * h^*(-\tau)|_{\tau=0}$

- Back to the white noise
  - When the input is a white noise with two-sided PSD $N_0/2$, the output PSD and the power of an LTI system with impulse response $h(t)$ is given, respectively, by

    $$S_{YY} = \frac{N_0}{2}|H(f)|^2 \text{ and } P_{YY} = \frac{N_0}{2}\int_{-\infty}^{\infty} |h(t)|^2 dt = \frac{N_0}{2}\int_{-\infty}^{\infty} |H(f)|^2 df.$$
  - A band-limited white noise is a filtered version of a white noise by an ideal LPF or an ideal BPF.