Complex Baseband Representation of Real-Valued CT Bandpass Random Process

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- Review. The transmitted signal is well modeled by **a real-valued continuous-time (CT) bandpass random process**. Moreover, the received signal contains a noise $N(t)$ often modeled by **a real-valued CT bandpass random process** called an additive white Gaussian noise (AWGN).
  - TD full/partial characterization of CT random processes
  - FD partial characterization of CT random processes
  - LTI filtering of CT random process

- Preview.
  - **complex envelope** of real-valued CT bandpass random process
  - @Tx: peak-to-average power ratio (PAPR) vs. peak-to-mean envelope power ratio (PMEPR) of real-valued CT bandpass random process
  - @Rx: output at I-Q demodulator

- Review. Given **a deterministic real-valued CT bandpass signal** $x(t)$, if we choose $f_c > 0$ and $\theta \in [0, 2\pi)$ then there exist two real-valued CT baseband signals $x_c(t)$ and $x_s(t)$ such that

$$x(t) = x_c(t)\cos(2\pi f_c t + \theta) - x_s(t)\sin(2\pi f_c t + \theta). \qquad (1)$$

  - $x_l(t) \triangleq x_c(t) + \mathrm{j}x_s(t)$ is called the **complex envelope** of $x(t)$.

  - envelope, magnitude, phase, amplitude, angle

  - $x_c(t)$ and $x_s(t)$ become different for a different choice of f_c and θ.
    * Any choice of f_c is okay but, if $x(t)$ is bandlimited to $|f| \in [f_1, f_2]$ with bandwidth $W \triangleq f_2 - f_1$, $f_c \geq (f_2 - f_1)/2 = W/2$ is necessary for an I-Q demodulator to recover $x_l(t)$.

- Q. Why do we need $f_c \geq W/2$ for an I-Q demodulator to recover $x_l(t)$?

– Fig.

– The LPFs are real-in real-out filters having passband $-B \leq f \leq B$.

– Thus, we need $f_c + f_1 \geq f_2 - f_c$ for the LPFs to filter out the lower sideband term.

– Choosing $f_c \in (f_2/2, f_2)$ is sufficient because f_1 can be set to zero.

– Choosing $f_c \approx (f_1 + f_2)/2$ is a common practice.

- Q. Can we do the same for **a real-valued CT bandpass random process** $X(t)$?

- Theorem. Given a real-valued CT bandpass random process $X(t)$, if we choose $f_c > 0$ and $\theta \in [0, 2\pi)$ then there exist two real-valued CT baseband random processes $X_c(t)$ and $X_s(t)$ such that

$$X(t) = X_c(t)\cos(2\pi f_c t + \theta) - X_s(t)\sin(2\pi f_c t + \theta).$$

  - Proof is omitted. This is straightforward because Eq. (1) holds for each sample path of $X(t)$.
  - $X_l(t) \triangleq X_c(t) + \mathrm{j}X_s(t)$ can be called the **complex envelope** of $X(t)$.
  - envelope, magnitude, phase, amplitude, angle
  - Any choice of f_c is okay but, if $X(t)$ is bandlimited to $|f| \in [f_1, f_2]$ with bandwidth $W \triangleq f_2 - f_1$, $f_c \geq (f_2 - f_1)/2 = W/2$ is necessary for an I-Q demodulator to recover $X_l(t)$.

- The complex envelopes of
  - a transmitted signal and
  - a received signal

  are both important in communication systems as shown below.

- Peak-to-Average Power Ratio (PAPR) vs. Peak-to-Mean Envelope Power Ratio (PMEPR)
  - The two terms are often used interchangeably but, precisely speaking, they are different.
  - Def. Given an real-valued $X(t)$, **the PAPR/PAR over the time interval** $t \in [T_1, T_2]$ is defined as

$$\text{PAPR} \triangleq \frac{\max\limits_{t\in[T_1,T_2]} (X(t))^2}{\frac{1}{T_2-T_1}\int_{T_1}^{T_2} \mathbb{E}\{(X(t))^2\}dt}$$

    * As the name says, it is the peak power divided by the average power.
      · It is a unitless quantity.
      · A low PAPR at a PA input is desirable in designing a power efficient transmitter. Thus, the lower the better.
      · A major disadvantage of OFDM is its high PAPR of about 12 dB.

* In general, the peak power in the numerator is a random variable.
  - The complementary cumulative distribution function (CCDF) of the PAPR defined by

$$1 - F(x) \triangleq \Pr(\text{PAPR} > x).$$

    is often drawn for comparison in a log-log plane.
  - For CCDF, the lefter the better.

* As the observation interval increases, the CCDF moves rightward because the peak power in the numerator increases.
* If $X(t)$ is WSS, then the average power in the denominator is not be a function of the interval.
* If $X(t)$ is periodic with period T, then taking $t \in (-\infty, \infty)$ is equivalent to taking $t \in [0, T)$.
* The square root of the PAPR is called the crest factor.

– Def. Given an real-valued $X(t)$, **the PMEPR over the time interval** $t \in [T_1, T_2]$ is defined as

$$\text{PMEPR} \triangleq \frac{\max\limits_{t\in[T_1,T_2]} |X_l(t)|^2}{\frac{1}{T_2-T_1}\int_{T_1}^{T_2} \mathbb{E}\{|X_l(t)|^2\}dt}$$

* As the name says, it is the peak power **of the complex envelope** divided by the mean envelope power, i.e., the average power **of the complex envelope**.
* It can be shown that

$$\text{PAPR} \approx 2 \cdot \text{PMEPR}$$

for sufficiently high f_c. Other definitions of the PAPR may result in different scaling factors, e.g., 1.

· Since $X(t) = |X_l(t)| \cos(2\pi f_c t + \angle X_l(t))$, we have $\max_{t\in[T_1,T_2]}(X(t))^2 \lesssim \max_{t\in[T_1,T_2]} |X_l(t)|^2$, for sufficiently high f_c.

· By definition of a complex envelope, we have $\int_{T_1}^{T_2} \mathbb{E}\{(X(t))^2\}dt \approx \frac{1}{2} \cdot \int_{T_1}^{T_2} \mathbb{E}\{|X_l(t)|^2\}dt$, for sufficiently high f_c.

· Therefore, the conclusion follows.

* **Often, a good approximation of the CCDF is easier for the PMEPR** than for the PAPR.
  - Extreme value theorems are useful in evaluating the CCDF of a PMEPR.
  - For a bandlimited Gaussian WSS random process with PSD $S_{XX}(f)$, the CCDF of the PMEPR is approximately given by

$$\exp\left(-\sqrt{\pi \ln(W_{\text{rms}}T)}W_{\text{rms}}Te^{-x}\right),$$

    where $T \gg W_{\text{rms}}$ is the length of the observation interval and W_{rms} is the root mean square (RMS) bandwidth of the real bandpass random process $X(t)$.

  - As the product of the RMS bandwidth and the observation time increases, the PMEPR moves rightward or worsens.

– RMS bandwidth of real-valued random process

* For real baseband random process, $W_{\text{rms}} \triangleq \sqrt{\dfrac{\int_{-\infty}^{\infty} f^2 S_{XX}(f)\,\mathrm{d}f}{\int_{-\infty}^{\infty} S_{XX}(f)\,\mathrm{d}f}}$.

* For real bandpass random process, $W_{\text{rms}} = 2\sqrt{\dfrac{\int_{0}^{\infty} (f-\overline{f})^2 S_{XX}(f)\,\mathrm{d}f}{\int_{0}^{\infty} S_{XX}(f)\,\mathrm{d}f}}$

where $\overline{f} = \dfrac{\int_{0}^{\infty} f S_{XX}(f)\,\mathrm{d}f}{\int_{0}^{\infty} S_{XX}(f)\,\mathrm{d}f}$.

- Rx front end and down-converter output
  - Figs.

  - Suppose that the signal to be down-converted is modeled by **a real-valued CT bandpass random process** $Y(t) = \tilde{X}(t) + I(t) + N(t)$, where
    * $\tilde{X}(t)$ is the channel filtered version of the signal transmitted by the Tx,
    * $I(t)$ is an interference, and
    * $N(t)$ is an additive Gaussian noise.

- Then, the down-converter output (whether it is from a homodyne receiver or a heterodyne receiver) can be modeled by **a complex baseband random process** $Y_l(t) = \tilde{X}_l(t) + I_l(t) + N_l(t)$, where
  * $\tilde{X}_l(t)$ is the complex envelope of $\tilde{X}(t)$,
  * $I_l(t)$ is the complex envelope of $I(t)$, and
  * $N_l(t)$ is the complex envelope of $N(t)$.

- Since the Tx and Rx RF chains perform mutually inverse operations, **we can regard the signal transmitted by the Tx and received by the Rx as a complex baseband signal.** However, be careful not to totally ignore hardware impairments in the chains such as
  - (@Tx) I-Q imbalance, nonlinear distortion such as saturation and clipping
  - (@Rx) interference and noise from image-band, carrier frequency offset (CFO), phase noise/jitter, DC leakage, effects of automatic gain control (AGC)