# *Complex Baseband Representation of Real-Valued Bandpass WSS Random Process: TD Characterization*

Prof. Joon Ho Cho
Department of Electrical Engineering, POSTECH

- Review. The received signal contains a noise $N(t)$ often modeled by **a real-valued bandpass wide-sense stationary (WSS) random process** called an additive white Gaussian noise (AWGN).
  - complex envelope of a real-valued CT bandpass random process

- Preview. TD characterization
  - real-valued CT bandpass **WSS** random process
    * mean
    * **auto- and cross-correlation functions** of in-phase and quadrature components
  - complex envelope of real-valued CT bandpass **WSS** random process
    * mean
    * **auto-correlation function**
    * **complementary/pseudo auto-correlation function** $\rightarrow$ The complex envelope is proper.

- Theorem. Let $X(t)$ be a real-valued bandpass WSS random process such that

$$X(t) = X_c(t)\cos(2\pi f_c t) - X_s(t)\sin(2\pi f_c t).$$

Then, $X_c(t)$ and $X_s(t)$ have the following 1st- and 2nd-order properties.

  – (mean processes)

$$\mathbb{E}\{X_c(t)\} = \mathbb{E}\{X_s(t)\} = 0, \forall t. \quad (1)$$

  – (auto-correlation functions) There exists a real even function $R_{X_cX_c}(\tau) = R_{X_cX_c}(-\tau), \forall\tau$ such that

$$\mathbb{E}\{X_c(t+\tau)X_c(t)\} = \mathbb{E}[X_s(t+\tau)X_s(t)] = R_{X_cX_c}(\tau), \forall t, \forall\tau. \quad (2)$$

  – (cross-correlation function) There exists a real odd function $R_{X_sX_c}(\tau) = -R_{X_sX_c}(-\tau), \forall\tau$ such that

$$\mathbb{E}\{X_s(t+\tau)X_c(t)\} = R_{X_sX_c}(\tau) = -R_{X_sX_c}(-\tau), \forall t, \forall\tau. \quad (3)$$

- In other words,
  - The in-phase and the quadrature components $X_c(t)$ and $X_s(t)$ are **mean-zero** random processes.
  - $X_c(t)$ and $X_s(t)$ are **WSS** random processes having the **same even-symmetrical autocorrelation** functions.
  - $X_c(t)$ and $X_s(t)$ are **jointly WSS** and have an **odd-symmetrical cross-correlation** function.
    * The odd symmetry implies $R_{X_sX_c}(0) = -R_{X_sX_c}(-0) = 0$, i.e., $\mathbb{E}\{X_s(t)X_c(t)\} = 0, \forall t$. $\leftarrow$ uncorrelated in-phase and quadrature components **when sampled at the same time instant**.
    * It does **not** imply that the in-phase and the quadrature components are uncorrelated random processes, i.e., it does **not** imply that $\mathbb{E}\{X_s(t+\tau)X_c(t)\} = 0, \forall t, \forall \tau$.

- Corollary. The complex envelope $X_l(t) \triangleq X_c(t) + \mathrm{j}X_s(t)$ has the following 1st- and 2nd-order properties.
  - (mean process)
$$\mathbb{E}\{X_l(t)\} = 0, \forall t. \tag{4}$$
  - (auto-correlation function)
$$\mathbb{E}\{X_l(t+\tau)X_l^*(t)\} = 2R_{X_cX_c}(\tau) + 2\mathrm{j}R_{X_sX_c}(\tau), \forall t, \forall \tau. \tag{5}$$
  - (complementary auto-correlation function) $X_l(t)$ is proper.
$$\mathbb{E}\{X_l(t+\tau)X_l(t)\} = 0, \forall t, \forall \tau. \tag{6}$$
- Proofs of Eqs. (4)–(6)
  - $\mathbb{E}\{X_l(t)\} =$
  - $\mathbb{E}\{X_l(t+\tau)X_l^*(t)\} =$
  - $\mathbb{E}\{X_l(t+\tau)X_l(t)\} =$
- It can be shown that Eqs. (4)–(6) imply Eqs. (1)–(3).

- Corollary. The real bandpass WSS random process $X(t)$ has the following 1st- and 2nd-order properties.
  - (mean process)
$$\mathbb{E}\{X(t)\} = 0, \forall t. \tag{7}$$
  - (auto-correlation function)
$$\begin{aligned}\mathbb{E}\{X(t+\tau)X(t)\} = {} & R_{X_cX_c}(\tau)\cos(2\pi f_c\tau) \\ & -R_{X_sX_c}(\tau)\sin(2\pi f_c\tau), \forall t, \forall \tau.\end{aligned} \tag{8}$$
- Proofs of Eqs. (7) and (8)
  - $\mathbb{E}\{X(t)\} =$
  - $\mathbb{E}\{X(t+\tau)X(t)\} =$
- It can be shown that Eqs. (7) and (8) imply Eqs. (1)–(3).

- Proof of Eq. (1).
  - Define $\mu_{X_c}(t) \triangleq \mathbb{E}\{X_c(t)\}$ and $\mu_{X_s}(t) \triangleq \mathbb{E}\{X_s(t)\}$.
    * Then, $\mathbb{E}\{X(t)\} = ... = \mu_{X_c}(t)\cos(2\pi f_c t) - \mu_{X_s}(t)\sin(2\pi f_c t)$.
    * Note that the PSD of $X(t)$ is the sum of the PSD of a deterministic signal $\mathbb{E}\{X(t)\}$ and the PSD of a zero-mean random process $X(t) - \mathbb{E}\{X(t)\}$.

    (**) Since $X(t)$ is real bandpass, $\mathbb{E}\{X(t)\}$ is also real bandpass, which implies $\mu_{X_c}(t)$ and $\mu_{X_s}(t)$ are real baseband signals.

  - Due to the wide-sense stationarity, there exists $c$ such that $\mathbb{E}\{X(t)\} = c, \forall t$.
    * However, $c$ cannot be non-zero because $X(t)$ is a bandpass random process that cannot have an impulse at $f = 0$.

    (***) Thus, a real bandpass signal $\mu_{X_c}(t)\cos(2\pi f_c t) - \mu_{X_s}(t)\sin(2\pi f_c t) = 0, \forall t$.

  - Therefore, by (**) and (***), the real baseband mean processes must satisfy $\mu_{X_c}(t) = \mu_{X_s}(t) = 0, \forall t$.

- Proofs of Eqs. (2) and (3).
  - Due to the wide-sense stationarity of $X(t)$, there exists $R_{XX}(\tau)$ such that

$$
\begin{aligned}
R_{XX}(\tau) &= \mathbb{E}\{X(t+\tau)X(t)\} \\
&= \mathbb{E}\left\{ \frac{X_l(t+\tau)e^{j2\pi f_c(t+\tau)} + X_l^*(t+\tau)e^{-j2\pi f_c(t+\tau)}}{2} \right. \\
&\qquad \left. \times \frac{X_l(t)e^{j2\pi f_c t} + X_l^*(t)e^{-j2\pi f_c t}}{2} \right\} \\
&= \ldots \\
&= \frac{1}{2}\mathrm{Re}\left[ \mathbb{E}\left\{X_l(t+\tau)X_l(t)\right\} e^{j2\pi f_c(2t+\tau)} \right] \\
&\qquad + \frac{1}{2}\mathrm{Re}\left[ \mathbb{E}\left\{X_l(t+\tau)X_l^*(t)\right\} e^{j2\pi f_c\tau} \right], \forall t, \forall \tau.
\end{aligned}
$$

  This implies that there exists a complex-valued function $R_{X_lX_l}(\tau)$ such that

$$
\mathbb{E}\{X_l(t+\tau)X_l(t)] = 0 \text{ and } \tag{9}
$$

$$
\mathbb{E}\{X_l(t+\tau)X_l^*(t)] = R_{X_lX_l}(\tau), \forall t, \forall \tau. \tag{10}
$$

– Note that

$$\begin{aligned}\mathbb{E}\{X_l(t+\tau)X_l(t)] =& \mathbb{E}\{(X_c(t+\tau)+\mathrm{j}X_s(t+\tau))(X_c(t)+\mathrm{j}X_s(t))] \\ =& \{R_{X_cX_c}(t+\tau,t)-R_{X_sX_s}(t+\tau,t)\} \\ &+\mathrm{j}\{R_{X_sX_c}(t+\tau,t)+R_{X_cX_s}(t+\tau,t)\}\,.\end{aligned}$$

Thus, (9) implies

$$R_{X_cX_c}(t+\tau,t)=R_{X_sX_s}(t+\tau,t) \tag{11}$$

$$R_{X_sX_c}(t+\tau,t)=-R_{X_cX_s}(t+\tau,t) \tag{12}$$

– Note also that

$$\begin{aligned}\mathbb{E}\{X_l(t+\tau)X_l^*(t)] =& \mathbb{E}\{(X_c(t+\tau)+\mathrm{j}X_s(t+\tau))(X_c(t)-\mathrm{j}X_s(t))] \\ =& \{R_{X_cX_c}(t+\tau,t)+R_{X_sX_s}(t+\tau,t)\} \\ &+\mathrm{j}\{R_{X_sX_c}(t+\tau,t)-\mathrm{j}R_{X_cX_s}(t+\tau,t)\}\,.\end{aligned}$$

Thus, (10), (11), and (12) imply

$$R_{X_lX_l}(\tau)=2R_{X_cX_c}(t+\tau,t)+2\mathrm{j}R_{X_sX_c}(t+\tau,t).$$

- Therefore, there exist real functions $R_{X_cX_c}(\tau)$ and $R_{X_sX_c}(\tau)$ such that

$$R_{X_cX_c}(\tau) = R_{X_cX_c}(t+\tau, t) = R_{X_sX_s}(t+\tau, t) \text{ and} \tag{13}$$
$$R_{X_sX_c}(\tau) = R_{X_sX_c}(t+\tau, t) = -R_{X_cX_s}(t+\tau, t). \tag{14}$$

  * Obviously, $R_{X_cX_c}(\tau)$ is an even function.
  * Since $-R_{X_cX_s}(t+\tau, t) = -R_{X_sX_c}(t, t+\tau) = -R_{X_sX_c}(-\tau)$, $R_{X_sX_c}(\tau)$ is an odd function.

- Remarks
  - In-phase and quadrature components
    * They are zero-mean jointly WSS random processes.
    * Their auto-correlation functions are identical and even symmetrical.
      · $\mathbb{E}\{(X_c(t))^2\} = R_{X_cX_c}(0) = R_{X_sX_s}(0) = \mathbb{E}\{(X_s(t))^2\}$.
    * Their cross-correlation function is an odd function.
      · $\mathbb{E}\{X_s(t)X_c(t)\} = R_{X_sX_c}(0) = 0$.

– Complex envelope

* It is a zero-mean proper-complex WSS random process.
* We define in this course the auto-correlation function of a complex-valued random process $Z(t)$ as

$$R_{ZZ}(t+\tau, t) \triangleq \mathbb{E}\{Z(t+\tau)Z^*(t)\}.$$

  · The autocorrelation function is conjugate symmetrical because

  $$R_{X_lX_l}(\tau) = \mathbb{E}[X_l(t+\tau)X_l^*(t)] = (\mathbb{E}[X_l(t)X_l^*(t+\tau)])^* = R^*_{X_lX_l}(-\tau)$$

  · $P_{X_lX_l} = \mathbb{E}\{|X_l(t)|^2\} = 2R_{X_cX_c}(0) = R_{X_cX_c}(0) + R_{X_sX_s}(0).$
* A complex-valued random process $Z(t)$ is called proper-complex if its complementary auto-covariance function vanishes, i.e.,

$$\mathbb{E}\{(Z(t+\tau) - \mathbb{E}[Z(t+\tau)])(Z(t) - \mathbb{E}[Z(t)])] = 0, \forall t, \forall \tau.$$

  · For a zero-mean complex random process, if its complementary auto-correlation function vanishes, it is proper.

  · Since $X_l(t)$ is zero-mean, the vanishing complementary auto-correlation function implies that $X_l(t)$ is proper.

* real bandpass process
    - It is a real bandpass WSS zero-mean random process
    - $P_{XX} = R_{XX}(0) = R_{X_cX_c}(0)\cos 0 - 2R_{X_sX_c}(0)\sin 0 = R_{X_cX_c}(0) = R_{X_sX_s}(0)$