Complex Baseband Representation of Real-Valued Bandpass WSS Random Process: FD Characterization

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- Review. The received signal contains a noise $N(t)$ often modeled by **a real-valued bandpass wide-sense stationary (WSS) random process** called an additive white Gaussian noise (AWGN).
  - TD characterization of in-phase and quadrature components of a real-valued CT bandpass WSS random process
  - TD characterization of complex envelope of a real-valued CT bandpass WSS random process

- Preview. FD characterization
  - real-valued CT bandpass **WSS** random process
    * **PSDs and xPSD** of in-phase and quadrature components
  - complex envelope of a real-valued CT bandpass **WSS** random process
    * zero-mean proper
    * **PSD** of complex envelope
  - TD and FD characterization of a real-valued CT bandpass **WSS Gaussian** random process

- Review. TD characterization of **real-valued bandpass WSS** random process
  - Let $X(t)$ be written as

$$X(t) = X_c(t)\cos(2\pi f_c t) - X_s(t)\sin(2\pi f_c t) = \mathrm{Re}\{X_l(t)e^{\mathrm{j}2\pi f_c t}\}.$$

  - (mean) $X(t), X_c(t), X_s(t)$, and $X_l(t)$ are **all zero-mean** random processes.

(i) $X_c(t)$ and $X_s(t)$ are **jointly WSS** random processes with **identical real-even auto**-correlation function $R_{X_cX_c}(\tau)$ and **real-odd cross**-correlation function $R_{X_sX_c}(\tau)$.

(ii) $X_l(t)$ is **proper**-complex WSS random process and has a **conjugate-symmetrical auto**-correlation function

$$R_{X_lX_l}(\tau) = 2R_{X_cX_c}(\tau) + 2\mathrm{j}R_{X_sX_c}(\tau).$$

(iii) $X(t)$ has auto-correlation function given by

$$R_{XX}(\tau) = R_{X_cX_c}(\tau)\cos(2\pi f_c\tau) - R_{X_sX_c}(\tau)\sin(2\pi f_c\tau).$$

- PSD and xPSD of in-phase and quadrature components
  - (From auto-correlation to PSD) real-even $R_{X_cX_c}(\tau) = R_{X_sX_s}(\tau)$ implies **real-even** non-negative $S_{X_cX_c}(f) = S_{X_sX_s}(f)$ where
$$R_{X_cX_c}(\tau) \stackrel{\mathcal{F}}{\leftrightarrow} S_{X_cX_c}(f).$$
  - (From cross-correlation to xPSD) real-odd $R_{X_sX_c}(\tau) = R_{X_cX_s}(-\tau)$ implies pure-imaginary odd $S_{X_sX_c}(f) = S_{X_cX_s}(-f)$ where
$$R_{X_sX_c}(\tau) \stackrel{\mathcal{F}}{\leftrightarrow} S_{X_sX_c}(f), \quad R_{X_sX_c}(-\tau) \stackrel{\mathcal{F}}{\leftrightarrow} S_{X_cX_s}(f).$$

In other words, it implies **real-odd** $\mathrm{j}S_{X_sX_c}(f) = \mathrm{j}S_{X_cX_s}(-f)$ where
$$\mathrm{j}R_{X_sX_c}(\tau) \stackrel{\mathcal{F}}{\leftrightarrow} \mathrm{j}S_{X_sX_c}(f), \quad \mathrm{j}R_{X_sX_c}(-\tau) \stackrel{\mathcal{F}}{\leftrightarrow} \mathrm{j}S_{X_cX_s}(f).$$

– (From $S_{XX}(f)$ to $S_{X_lX_l}(f)$)

$$S_{X_lX_l}(f) = \text{LPF}\{4S_{XX}(f + f_c)\}.$$

– (From $S_{X_lX_l}(f)$ to $S_{XX}(f)$)

$$S_{XX}(f) = \frac{S_{X_lX_l}(f - f_c) + S_{X_lX_l}(-(f + f_c))}{4}$$

is a non-negative even function.

- Recall that $X_l(t) = X_c(t) + \mathrm{j}X_s(t)$.
- (From $S_{X_cX_c}(f), S_{X_sX_c}(f)$ to $S_{X_lX_l}(f)$)

$$
\begin{aligned}
S_{X_lX_l}(f) &= ... = (S_{X_cX_c}(f) + S_{X_sX_s}(f)) + \mathrm{j}(S_{X_sX_c}(f) - S_{X_cX_s}(f)) \\
&= 2S_{X_cX_c}(f) + 2\mathrm{j}S_{X_sX_c}(f)
\end{aligned}
$$

is a real non-negative function.

- (From $S_{X_lX_l}(f)$ to $S_{X_cX_c}(f)$ and $\mathrm{j}S_{X_sX_c}(f)$)

$$2S_{X_cX_c}(f) = \frac{S_{X_lX_l}(f) + S_{X_lX_l}(-f)}{2}$$

is the **even part** of $S_{X_lX_l}(f)$. Recall that $P_{X_lX_l} = P_{X_cX_c} + P_{X_sX_s} = 2P_{X_cX_c}$.

$$2\mathrm{j}S_{X_sX_c}(f) = \frac{S_{X_lX_l}(f) - S_{X_lX_l}(-f)}{2}$$

is the **odd part** of $S_{X_lX_l}(f)$.

– Recall that

$$R_{XX}(\tau) = R_{X_cX_c}(\tau)\cos(2\pi f_c\tau) - R_{X_sX_c}(\tau)\sin(2\pi f_c\tau),$$

where $R_{XX}(\tau)$ can be viewed as a real-valued bandpass signal with the in-phase component $R_{X_cX_c}(\tau)$ and the quadrature component $R_{X_sX_c}(\tau)$.

– (From $S_{X_cX_c}(f)$ and $\mathrm{j}S_{X_sX_c}(f)$ to $S_{XX}(f)$)

$$\begin{aligned} S_{XX}(f) &= \ldots \\ &= \frac{S_{X_cX_c}(f-f_c) + \mathrm{j}S_{X_sX_c}(f-f_c)}{2} \\ &\quad + \frac{S_{X_cX_c}(-(f+f_c)) + \mathrm{j}S_{X_sX_c}(-(f+f_c))}{2} \end{aligned}$$

is a non-negative even function.

– (From $S_{XX}(f)$ to $S_{X_cX_c}(f)$ and $\mathrm{j}S_{X_sX_c}(f)$)

$$S_{X_cX_c}(f) = \ldots$$

is twice the even part of .... Recall that $P_{XX} = P_{X_cX_c}$.

$$\mathrm{j}S_{X_sX_c}(f) = \ldots$$

is twice the odd part of ....

- Definition. A real-valued CT random process $X(t)$ is Gaussian if,

  $\forall N \in \mathbb{N}$ and

  $\forall (t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$,

  the random vector

$$\begin{bmatrix} X(t_1) \\ X(t_2) \\ \vdots \\ X(t_N) \end{bmatrix}$$

  is a real-valued Gaussian random vector.

- Lemma. **The mean process $\mu_X(t) \triangleq \mathbb{E}\{X(t)\}$ and the auto-correlation function $R_{XX}(t+\tau, t) \triangleq \mathbb{E}\{X(t+\tau)X(t)\}$ fully characterize a real-valued Gaussian random process.**
  - Proof. Define $\underline{X}^T \triangleq [X(t_1)\ \ X(t_2)\ \ \ldots\ \ X(t_N)]^T$. Then, its joint characteristic function fully characterizes $\underline{X}$ and can be written as

$$\Phi_{X(t_1),X(t_2),\cdots,X(t_N)}(\underline{\omega}) = \mathbb{E}\left\{e^{\mathrm{j}\underline{\omega}^T\underline{X}}\right\} = \exp\left(\mathrm{j}\underline{\omega}^T\underline{\mu} - \frac{\underline{\omega}^T C\underline{\omega}}{2}\right),$$

where the mean vector $\underline{\mu}$ with $[\underline{\mu}]_i = \mathbb{E}[X(t_i)]$ and the covariance matrix C with

$$[C]_{i,j} = \mathbb{E}\{(X(t_i) - \overline{X}(t_i))(X(t_j) - \overline{X}(t_j))\}$$
$$= C_{XX}(t_i, t_j) = R_{XX}(t_i, t_j) - \mu_X(t_i)\mu_X(t_j)$$

can be found from the mean process $\mu_X(t)$ and the auto-correlation function $R_{XX}(t+\tau, t)$.

– This can be done for $\forall N \in \mathbb{N}$ and $\forall (t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$.

– Therefore, by Kolmogorov's extension theorem, the conclusion follows.

- Theorem. If a real-valued random process $X(t)$ is **Gaussian and WSS**, then it is **SSS**.

  – Proof. By the definition of wide-sense stationarity, $\mu_X(t) = \mu_X, \forall t$ and $R_{XX}(t+\tau, t) = R_{XX}(\tau)$. Define $\underline{X}^T \triangleq [X(t_1)\ X(t_2)\ \ldots\ X(t_N)]^T$ and $\underline{Y}^T \triangleq [X(t_1+\Delta)\ X(t_2+\Delta)\ \ldots\ X(t_N+\Delta)]^T$.

  Then, $\mathbb{E}\{\underline{X}\} = \mathbb{E}\{\underline{Y}\} = \mu \underline{1}_N$. Moreover, $[C_{\underline{XX}}]_{i,j} = C_{XX}(t_i - t_j) = [C_{\underline{YY}}]_{i,j} = C_{XX}(t_i - \Delta - (t_j - \Delta)) = R_{XX}(t_i - t_j) - \mu^2$.

Thus, the joint characteristic functions of $\underline{X}$ and $\underline{Y}$ are identical regardless of Δ.

– This can be done for $\forall N \in \mathbb{N}$ and $\forall (t_1, t_2, \cdots, t_N) \in \mathbb{R}^N$.

– Therefore, by the definition of strict-sense stationarity, the conclusion follows.

- Corollary. If a real-valued random process $X(t)$ is **WSS zero-mean Gaussian**, then it is fully characterized by its PSD.
  – Proof. Since it is of zero-mean, the characteristic function is determined only by the auto-correlation function that is equivalent to the PSD. Therefore, the conclusion follows from the theorem.

- Corollary. If a real-valued random process $X(t)$ is **bandpass WSS Gaussian**, then it is fully characterized by its PSD.
  – Since it is a bandpass WSS random process, the mean function is zero. Therefore, the conclusion follows from the above corollary.

- Q. What can we say about the in-phase and the quadrature components of a real-valued **bandpass WSS Gaussian** random process $X(t)$?
- A. $X_c(t)$ and $X_s(t)$ are jointly WSS identically distributed zero-mean Gaussian random processes.
  - $X_c(t)$ and $X_s(t)$ are jointly Gaussian because any collection of samples is a linear transform of samples from the Gaussian random process.
  - $X_c(t)$ and $X_s(t)$ are zero-mean processes because ...
  - $X_c(t)$ and $X_s(t)$ have identical real-even auto-correlation functions and non-negative symmetrical PSDs because ...
  - $X_c(t)$ and $X_s(t)$ have real-odd cross-correlation function and pure imaginary odd xPSD.
- Remarks

– If $X(t)$ is a real-valued **bandpass WSS Gaussian** random process with power $P_{XX} = \sigma^2$, then

$$f_{X_c(t),X_s(t)}(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2 + y^2}{2\sigma^2}\right)$$

because ... Thus, for each t, we have i.i.d. zero-mean circularly symmetric 2-D Gaussian random vectors $[X_c(t)\ X_s(t)]^T$.

– If sampled at $t_1 \neq t_2$, then $X_c(t_1)$ and $X_s(t_2)$ are still identically distributed Gaussian random variables with mean zero and variance σ^2, but not necessarily independent. Thus, $f_{X_c(t_1),X_s(t_2)}(x, y)$ is 2-D Gaussian but no longer circularly symmetric in general.

- Q. What can we say about the complex envelope of a real-valued **bandpass WSS Gaussian** random process $X(t)$?
- A. $X_l(t)$ is a WSS zero-mean proper-complex Gaussian random process.
  – $X_l(t)$ is a complex Gaussian random process because ...
  – $X_l(t)$ is zero-mean WSS because ...
  – $X_l(t)$ is proper because ...