Lec. 08

2013년 3월 21일 목요일
오후 1:51

Review

- 4 modes of the convergence of a sequence of random variables $X_1, X_2, X_3, \cdots$ to a random variable X

strong LLN

(i) $X_n \to X$ a.s.

$$P(\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}) = 1$$

(ii) $X_n \to X$ in the p-th mean. $1 \le p$

$$E\{|X_n - X|^p\} \to 0 \quad \text{as } n \to \infty$$

weak... LLN

(iii) $X_n \to X$ in prob

$$\forall \varepsilon > 0, \quad \Pr(|X_n - X| > \varepsilon) \to 0 \text{ as } n \to \infty$$

(iv) $X_n \to X$ in distribution

CTL

$$F_{X_n}(x) \to F_X(x) \quad \text{everywhere}$$

$$\Phi_{X_n}(\omega) \to \Phi_X(\omega) \quad \text{everywhere}$$

$$\text{a.s.} \Rightarrow \text{in p.} \Leftarrow L^p$$

$$\text{in p.} \Downarrow \text{in distribution}$$

$Pr(|X-\overline{X}|>k\sigma) \le \frac{1}{k^2}$

Page 01

2013년 3월 21일 목요일

오후 1:52

- The weak law of large numbers

  Given uncorrelated r.v.'s X_n w/ $E\{X_n\}=\mu$ and $Var\{X_n\}=\sigma^2$ $\forall n$,

$$\frac{X_1+X_2+\cdots+X_n}{n} \to \mu \text{ in probability.}$$

- Implication

  $\forall \varepsilon>0$,

$$Pr\left(\left|\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right|>\varepsilon\right) \to 0 \text{ as } n\to\infty$$

$\Leftrightarrow \forall \varepsilon>0$,

$\forall \varepsilon'>0$, $\exists N(\varepsilon') \in \mathbb{N}$ s.t.

$$n \ge N(\varepsilon') \Rightarrow Pr\left(\left|\frac{X_1+X_2+\cdots+X_n}{n}-\mu\right|>\varepsilon\right) < \varepsilon'.$$

$\Rightarrow \forall \varepsilon>0$, $\exists N(\varepsilon) \in \mathbb{N}$ s.t.

$$Pr\left(\left|\frac{X_1+\cdots+X_n}{n}-\mu\right|>\varepsilon\right) < \varepsilon$$

for sufficiently large n.

- AEP

$-\log p(X_n)$: i.i.d. random variables

$E\{-\log p(X_n)\} = H(X)$

Page 02

2013년 3월 21일 목요일

오후 1:52

$$-\frac{1}{n}\log p(X_1, X_2, \cdots, X_n)$$

By the WLLN,

$$\frac{-\log p(X_1) - \log p(X_2) \cdots - \log p(X_n)}{n} \to H(X)$$

in prob.

- $\Pr(X > \varepsilon) < \varepsilon$

$\Leftrightarrow 1 - P(X \leq \varepsilon) < \varepsilon$

$\Leftrightarrow P(X \leq \varepsilon) > 1 - \varepsilon.$

- $\Pr\left(\left|-\frac{1}{n}\log p(X_1, X_2, \cdots, X_n) - H(X)\right| \leq \varepsilon\right)$

$= P\left(\boxed{\left\{(x_1, \cdots, x_n) \in \mathcal{X}^n : \left|-\frac{1}{n}\log p(x_1, x_2, \cdots, x_n) - H(X)\right| \leq \varepsilon\right\}}\right)$

$A_\varepsilon^{(n)} \triangleq$

$\underbrace{2^{-n(H(X)+\varepsilon)}} \leq p(x_1, \cdots, x_n) \leq 2^{-n(H(X)-\varepsilon)}$

- Properties of $A_\varepsilon^{(n)}$

(i) If $(x_1, \cdots, x_n) \in A_\varepsilon^{(n)}$

$$H(X) - \varepsilon \leq -\frac{1}{n}\log p(x_1, x_2, \cdots, x_n) \leq H(X) + \varepsilon$$

(ii) $\Pr((X_1, X_2, \cdots, X_n) \in A_\varepsilon^{(n)}) > 1 - \varepsilon$, for sufficiently large n.

(iii) $|A_\varepsilon^{(n)}| \leq 2^{n(H(X)+\varepsilon)}$

(iv) $|A_\varepsilon^{(n)}| \geq (1-\delta)2^{n(H(X)-\varepsilon)}$ for s. l. n

(iv) $|A_\varepsilon^{(n)}| > (1-\varepsilon)2^{n(H(X)-\varepsilon)}$ for s.l. n

- proof of (iii) $|A_\varepsilon^{(n)}| \le 2^{n(H(X)+\varepsilon)}$

$$\sum_{\underline{x}\in \mathcal{X}^n} p(\underline{x}) = 1 \ge \underbrace{\sum_{\underline{x}\in A_\varepsilon^{(n)}} p(\underline{x})}_{P(A_\varepsilon^{(n)})}$$

$\ge 2^{-n(H(X)+\varepsilon)}$

$$\ge \sum_{\underline{x}\in A_\varepsilon^{(n)}} 2^{-n(H(X)+\varepsilon)}$$

$$= |A_\varepsilon^{(n)}|\, 2^{-n(H(X)+\varepsilon)}$$

$$2^{n(H(X)+\varepsilon)} \ge |A_\varepsilon^{(n)}|$$

- Proof of property (iv)

By property (ii), $P(A_\varepsilon^{(n)}) > 1-\varepsilon$ for s.l.n

$$1-\varepsilon < P(A_\varepsilon^{(n)}) = \sum_{\underline{x}\in A_\varepsilon^{(n)}} p(\underline{x})$$

$\le 2^{-n(H(X)-\varepsilon)}$

for s.l.n

$$\le \sum_{\underline{x}\in A_\varepsilon^{(n)}} 2^{-n(H(X)-\varepsilon)}$$

$$= |A_\varepsilon^{(n)}|\, 2^{-n(H(X)-\varepsilon)}$$

$$\therefore \quad |A_\varepsilon^{(n)}| > (1-\varepsilon)\, 2^{n(H(X)-\varepsilon)} \text{ for s.l.n.}$$

- **Achievability**

Proof by construction

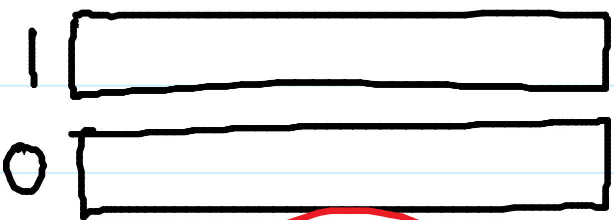
Consider $(\varepsilon_k)_{k=1}^{\infty}$, $1 > \varepsilon_k \searrow 0$.

Then, for each ε_k, there exists n_k s.t.

$P(A_{\varepsilon_k}^{(n_k)}) > 1-\varepsilon_k$, which implies

$$(1-\varepsilon_k)2^{n_k(H(X)-\varepsilon_k)} < |A_{\varepsilon_k}^{(n_k)}| \le 2^{n_k(H(X)+\varepsilon_k)}$$

We assign $n_k(H(X)+\varepsilon_k)+2$ bits to distinctively describe the members of $A_{\varepsilon_k}^{(n_k)}$



Code rate R_k [codeword symbols/source symbol]

$$= \frac{n_k(H(X)+\varepsilon_k)+2}{n_k} \to H(X)$$

Error rate $P_e^{(n_k)} = 1 - P(A_{\varepsilon_k}^{(n_k)}) < \varepsilon_k \to 0$

- **Converse**

$A \Leftrightarrow B$ (achievability)

$\Rightarrow$ Direct part

$\Leftarrow$ converse