Lec. 11

2013년 4월 3일 수요일

오후 9:08

- Review

Shannon's information measures

$$H(X) \triangleq E\left\{\log \frac{1}{P_X(X)}\right\} = \sum_{i=1}^{|X|} p_i \log \frac{1}{p_i}$$

$$H(X,Y) = E\left\{\log \frac{1}{P_{X,Y}(X,Y)}\right\}$$

$$\vdots$$

$$H(X_1, X_2, \cdots, X_N) = E\left\{\log \frac{1}{P_{X_1,X_2,\cdots,X_N}(X_1,X_2,\cdots,X_N)}\right\}$$

$$H(X|Y=y) = E\left\{\log \frac{1}{P_{X|Y}(X|Y)} \boxed{\,\Big|\, Y=y}\right\}$$

$$H(X|Y) = E\left[E\left\{\log \frac{1}{P_{X|Y}(X|Y)} \,\Big|\, Y\right\}\right]$$

cf) $E[X|Y=y]$ ← determinisitc

$E[X|Y]$ ← r.v. ← ft.

$$= E\left\{\log \frac{1}{P_{X|Y}(X|Y)}\right\}$$

$$H(X,Y) = H(X) + H(Y|X)$$

$$= H(Y) + H(X|Y)$$

$$H(X_1,\cdots,X_N) = H(X_1) + H(X_2|X_1) + H(X_3|X_1,X_2) + \cdots + H(X_N|X_1,X_2,\cdots,X_{N-1})$$

$$I(X;Y) \triangleq H(X) - H(X|Y) = H(Y) - H(Y|X)$$

- $I(X;Y) = H(X) - H(X|Y) = E\left\{ \log \frac{P_{X|Y}(X|Y)}{P_X(X)} \right\}$

$\parallel$

$I(Y;X) = H(Y) - H(Y|X) = E\left\{ \log \frac{P_{Y|X}(Y|X)}{P_Y(Y)} \right\}$

representation.

$= H(X) + H(Y) - H(X,Y) = E\left\{ \log \frac{P_{X,Y}(X,Y)}{P_X(X)P_Y(Y)} \right\}$

Q1. $I(X;X) = H(X)$

Q2. $I(X;Y) = 0$ iff X & Y are independent.

- $I((X,Y);Z) \triangleq H(X,Y) - H(X,Y|Z)$

$\triangleq H(Z) - H(Z|X,Y)$

$\triangleq H(X,Y) + H(Z) - H(X,Y,Z)$

$= E\left\{ \log \frac{P_{X,Y,Z}(X,Y,Z)}{P_{X,Y}(X,Y)\,P_Z(Z)} \right\}$

- $I(X;Y|Z=z) = E\left\{ \log \frac{P_{X,Y|Z}(X,Y|z)}{P_{X|Z}(X|z)\,P_{Y|Z}(Y|z)} \,\middle|\, Z=z \right\}$

- $I(X;Y|Z) \triangleq E\left\{ \log \frac{P_{X,Y|Z}(X,Y|Z)}{P_{X|Z}(X|Z)\,P_{Y|Z}(Y|Z)} \right\}$

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$$= E\left\{\log \frac{P_{X|Y,Z}(X|Y,Z)}{P_{X|Z}(X|Z)}\right\}$$

- $I(X;Y|Z) = H(X|Z) - H(X|Y,Z)$

$$= H(Y|Z) - H(Y|X,Z)$$

$$= E\left\{\log \frac{P_{Y|X,Z}(Y|X,Z)}{P_{Y|Z}(Y|Z)}\right\}$$

- $I(X_1, X_2, \cdots, X_N; Y) = H(X_1, X_2, \cdots, X_N) - H(X_1, X_2, \cdots, X_N|Y)$

$$= H(X_1) + H(X_2|X_1) + \cdots + H(X_N|X_1, X_2, \cdots, X_{N-1}) - H(X_1|Y) - H(X_2|X_1,Y) - H(X_N|X_1, \cdots, X_{N-1}, Y)$$

$$= I(X_1;Y) + I(X_2;Y|X_1) \cdots + I(X_N;Y|X_1, \cdots, X_{N-1})$$

where the equality holds

- $I(X;Y) \geq 0$ iff X & Y are independent.
  $I(X;Y|Z) \geq 0$ iff X & Y " conditionally " given $\{Z=z\}$

- The KL distance $D(p \| q)$
  Kullback Leibler
  $\equiv$ the information divergence
  $\equiv$ the relative entropy

- A distance metric $d(x,y)$
  (i) non-negativity
  $d(x,y) \geq 0$ where equality holds iff $x=y$
  (ii) commutativity
  $d(x,y) = d(y,x)$
  (iii) triangle inequality
  $d(x,y) \leq d(x,z) + d(z,y)$

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- Given two random variables

$$X \sim p_X(x)$$

$$Y \sim p_Y(y)$$

we call

$$D(p_X \| p_Y) \triangleq E\left\{\log \frac{p_X(X)}{p_Y(X)}\right\}$$

the KL distance from p_X to p_Y.

- Since

$$D(p_X \| p_Y) = \sum_{x \in \mathcal{X}} p_X(x) \log \frac{p_X(x)}{p_Y(x)}$$

$$= \sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_Y(x)} \quad \text{mismatched.}$$

$$- \sum_{x \in \mathcal{X}} p_X(x) \log \frac{1}{p_X(x)} \quad \text{matched}$$

- Physical meaning : mismatched codebook.
  - matched ".

Too good to be true!

$D(p_X \| p_Y) \geq 0$ where the equality holds iff $p_X(x) = p_Y(x), \forall x$

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- $$D(P_X \| P_Y) = E\left\{\log \frac{P_X(X)}{P_Y(X)}\right\}$$

$$= \sum_{x\in\mathcal{X}} P_X(x) \log \frac{P_X(x)}{P_Y(x)}$$

$$= \log e \sum_{x\in\mathcal{X}} P_X(x) \underbrace{\ln \frac{P_X(x)}{P_Y(x)}}_{\geq 1 - \frac{P_Y(x)}{P_X(x)}}$$

$$\log_a b = \frac{\log_c b}{\log_c a}$$

$$\log_a e \ln b = \log_a b$$

$$\geq \log e \sum_{x\in\mathcal{X}} P_X(x) \left(1 - \frac{P_Y(x)}{P_X(x)}\right)$$

$$= \log e \left(\sum_{x\in\mathcal{X}} P_X(x) - \sum_{x\in\mathcal{X}} P_Y(x)\right)$$

$$\geq \log e \,(1 - 1)$$

$$= 0$$

= iff $P_X(x) = P_Y(x), \forall x$.

- $$\boxed{D(P_X \| P_Y) = E\left\{\log \frac{P_X(X)}{P_Y(X)}\right\} = \sum_{x\in\mathcal{X}} P_X(x) \log \frac{P_X(x)}{P_Y(x)}} \neq \boxed{D(P_Y \| P_X) = E\left\{\log \frac{P_Y(Y)}{P_X(Y)}\right\} = \sum_{y\in\mathcal{Y}} P_Y(y) \log \frac{P_Y(y)}{P_X(y)}}$$

in general.

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- $D(P_{X,Y} \| P_{Z,W}) = E\left\{\log \frac{P_{X,Y}(X,Y)}{P_{Z,W}(X,Y)}\right\}$

$= E\left\{\log \frac{P_X(X) P_{Y|X}(Y|X)}{P_Z(X) P_{W|Z}(Y|X)}\right\}$

$= E\left\{\log \frac{P_X(X)}{P_Z(X)}\right\} + E\left\{\log \frac{P_{Y|X}(Y|X)}{P_{W|Z}(Y|X)}\right\}$

$\stackrel{\Delta}{=} D(P_X \| P_Z) + D(P_{Y|X} \| P_{W|Z})$

- $I(X;Y) = H(X) + H(Y) - H(X,Y)$

$= E\left\{\log \frac{P_{X,Y}(X,Y)}{P_X(X) P_Y(Y)}\right\}$

$= D(P_{X,Y} \| P_X \cdot P_Y)$

≥ 0

where $=$ holds iff X & Y are independent.

- $H(X) \geq H(X|Y)$, $H(Y) \geq H(Y|X)$ upper bound
- $H(X_1, X_2, \cdots, X_N) \leq \sum^{N} H(X_n)$ Independence