Lec. 18

2013년 4월 24일 수요일

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- Review

- Achievability of rate R:
  There exists a sequence of $(\lceil 2^{n_k R} \rceil, n_k)$ codes such that λ_k ($\triangleq$ maximal code-word error probability) $\to 0$.

- The supremum of achievable rates = operational ch. capa.
  $= \max_{P_X(x)} I(X;Y)$ = informational channel capa.

  "The channel coding theorem for DMC."

- Direct Part
  If $R \le \max_{P_X(x)} I(X;Y)$, then R is achievable.

- Converse
  If R is achievable, then $R \le \max_{P_X(x)} I(X;Y)$.

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- Achievability

(i) Consider the random $(\lceil 2^{nR} \rceil, n)$ coding w/ a certain $P_X(x)$

(ii) $\Pr(\hat{W} \neq m_i ; W = m_i)$ of the typical set decoder is given by

$$\Pr(\hat{W} \neq m_1 ; W = m_1) = \sum_{\underline{x}^{(n)} \in \mathcal{X}^n} \Pr(\hat{W} \neq m_1 , \underline{X}_1^{(n)} = \underline{x}^{(n)} ; W = m_1)$$

$$= \sum_{\underline{x}^{(n)} \in \mathcal{X}^n} \Pr(\hat{W} \neq m_1 \mid \underline{X}_1^{(n)} = \underline{x}^{(n)}, W = m_1) \times \Pr(\underline{X}_1^{(n)} = \underline{x}^{(n)} ; \cancel{W = m_1})$$

$$\leq \Pr(\{(\underline{X}^{(n)}, \underline{Y}^{(n)}) \notin A_\varepsilon^{(n)}\}) + (\lceil 2^{nR} \rceil - 1) \Pr(\{(\underline{\tilde{X}}^{(n)}, \underline{Y}^{(n)}) \in A_\varepsilon^{(n)}\})$$

$$\leq \varepsilon + 2^{nR} \, 2^{-n(I(X;Y) - 3\varepsilon)}$$

$$= \varepsilon + 2^{-n(I(X;Y) - R - 3\varepsilon)}$$

$\therefore \Pr(\hat{W} \neq m_i ; W = m_i) \to 0$ as $n \to \infty$

if $R < I(X;Y)$

(iii) Now choose $P_X(x) = \underset{P_X(x)}{\operatorname{argmax}} \, I(X;Y)$

& repeat steps (i) & (ii), then we can say

If $R < \max_{p_X(x)} I(X;Y)$ is used, (random coding & typical set decoding)
then

$$Pr(\hat{W} \neq m_i \,;\, W = m_i) \to 0 \quad \text{as} \quad n \to \infty.$$

(iv) Consider random bits w/ uniform distribution. Define

$$P_e(\mathcal{C}_n) \triangleq Pr(\hat{W} \neq W \,;\, \mathcal{C}_n)$$

↑ average codeword error prob. given $\mathcal{C}_n$.

$$\begin{aligned}
\bar{P}_{e,n} &= Pr(\hat{W} \neq W) \\
&= \sum_{\mathcal{C}_n} Pr(\hat{W} \neq W \mid \mathcal{C}_n)\, Pr(\mathcal{C}_n) \\
&= \sum_{\mathcal{C}_n} \left(\sum_{i=1}^{\lceil 2^{nR} \rceil} Pr(\hat{W} \neq W \mid W = m_i, \mathcal{C}_n)\, Pr(W = m_i \mid \cancel{\mathcal{C}_n}) \right) \cdot Pr(\mathcal{C}_n) \\
&= \sum_{i=1}^{\lceil 2^{nR} \rceil} \left[\sum_{\mathcal{C}_n} Pr(\hat{W} \neq m_i \mid W = m_i, \mathcal{C}_n)\, Pr(\mathcal{C}_n \mid W = m_i) \right] Pr(W = m_i)
\end{aligned}$$

$\lceil 2^{nR} \rceil$

$$= \sum_{i=1} Pr(\hat{W} \neq m_i \mid W = m_i)\, Pr(W = m_i)$$

$$\leq \sum_{i=1}^{\lceil 2^{nR} \rceil} \left(\varepsilon + 2^{-n(I(X;Y)-R-3\varepsilon)} \right) \Pr(W=m_i)$$

$$= \varepsilon + 2^{-n(I(X;Y)-R-3\varepsilon)}$$

If bits are i.i.d. uniform r.v.'s,

the random coding w/ $\arg\max_{p_X(x)} I(X;Y)$,

& the typical set decoder are used,

then $R < \max_{p_X(x)} I(X;Y)$

$\overline{P}_{e,n} < \varepsilon'$ for s.l. n.

$\Rightarrow$ There exists $\mathcal{C}_n'$ s.t. $P_e(\mathcal{C}_n') < \varepsilon'$ for s.l. n.

(when bits are i.i.d. uniform)

(v) Using $\mathcal{C}_n'$,

$$\left(\begin{array}{l} \Pr(\hat{W} \neq m_1 ; W = m_1) \\ \Pr(\hat{W} \neq m_2 ; W = m_2) \\ \quad \vdots \\ \Pr(\hat{W} \neq m_{\lceil 2^{nR} \rceil} ; W = m_{\lceil 2^{nR} \rceil}) \end{array} \right.$$

Among these $\lceil 2^{nR} \rceil$ conditional codeword error probabilities, we can find

$\lceil 2^{nR} \rceil$ ... s.t. $< 2\varepsilon'$

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$$\log \left\lfloor \frac{\lceil 2^{nR} \rceil}{2} \right\rfloor / n = \frac{nR-1}{n} = R - \frac{1}{n} \to R.$$

We know have $\tilde{\mathcal{C}}_n$, $\lambda_n(\tilde{\mathcal{C}}_n) < 2\varepsilon'$ for s.l. n

(vi) Choose $(\varepsilon_k')_k$ s.t. $\varepsilon_k' \to 0$ as $k \to \infty$.

Then, $\exists (\tilde{\mathcal{C}}_{n_k})_k$ s.t.

$$\lambda_{n_k}(\tilde{\mathcal{C}}_{n_k}) < 2\varepsilon_k'$$

$\therefore R \leq \max_{P_X(x)} I(X;Y)$ is achievable.

- Converse.

Recall that

$$H(W|\hat{W}) \leq h_b(P_e) + P_e \cdot \log(\lceil 2^{nR} \rceil - 1)$$

E : error indicator function of $w, \hat{w}$.

" $H(E|W,\hat{W}) = 0$ "

To show,

" Suppose there exists a sequence of codes $\mathcal{C}_k$ s.t. $R_k \to R$

$$\lambda_k \to 0.$$

Then, $R \leq \max_{P_X(x)} I(X;Y)$. "

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Notice that $\underline{Y}^{(n_k)} \to \boxed{\mathrm{dec}^{(k)}} \to \hat{W}$

$\cancel{H(W|\hat{W})}$

$$H(W|\underline{Y}^{(n)}) = H(W|\underline{Y}^{(n)}) + \underbrace{H(E|W,\underline{Y}^{(n)})}_{=0}$$

$$= H(W,E|\underline{Y}^{(n)})$$

$$= \underbrace{H(E|\underline{Y}^{(n)})}_{\le 1} + \underbrace{H(W|E,\underline{Y}^{(n)})}_{\substack{\le Pe\cdot\log(\lceil 2^{nR}\rceil - 1) \\ \le nR\cdot Pe}}$$

where

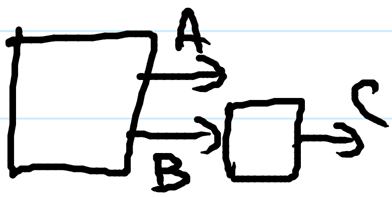
$$H(W|E,\underline{Y}^{(n)}) = \sum_{e=0}^{1}\sum_{\underline{y}^{(n)}} H(W|E=e,\underline{Y}^{(n)}=\underline{y}^{(n)})\,Pr(E=e,\underline{Y}^{(n)}=\underline{y}^{(n)})$$

$$= \sum_{\underline{y}^{(n)}} \cancel{H(W|E=0,\underline{Y}^{(n)}=\underline{y}^{(n)})}\,Pr(E=0,\underline{Y}^{(n)}=\underline{y}^{(n)}) + H(W|E=1,\underline{Y}^{(n)}=\underline{y}^{(n)})\,Pr(E=1,\underline{Y}^{(n)}=\underline{y}^{(n)})$$

$$\le \sum_{\underline{y}^{(n)}} \log(\lceil 2^{nR}\rceil - 1)\,Pr(E=1,\underline{Y}^{(n)}=\underline{y}^{(n)})$$

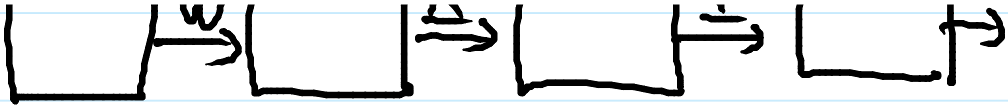
$$= Pe\cdot\log(\lceil 2^{nR}\rceil - 1)$$

$$\le nR\cdot Pe$$



Notice $\boxed{} \xrightarrow{W} \boxed{} \xrightarrow{\underline{X}^{(n)}} \boxed{} \xrightarrow{\underline{Y}^{(n)}} \boxed{} \to \hat{W}$

Notice



$$(\because W \to X^{(n)} \to Y^{(n)} \ \& \ Y^{(n)} \to W \to X^{(n)})$$

$$H(W) = H(W \mid Y^{(n)}) + I(W; Y^{(n)})$$

$$= H(W \mid Y^{(n)}) + I(X^{(n)}; Y^{(n)})$$

$$\leq 1 + nR \cdot P_e + H(Y^{(n)}) - H(Y^{(n)} \mid X^{(n)})$$

$$\leq 1 + nR \cdot P_e + \sum_{i=1}^{n} H(Y_i^{(n)}) - \underbrace{H(Y^{(n)} \mid X^{(n)})}_{\sum_{i=1}^{n} H(Y_i^{(n)} \mid Y_1^{(n)}, \ldots, Y_{i-1}^{(n)}, X^{(n)})}$$

$$\to X_i^{(n)}$$

$$= 1 + nR \cdot P_e + \sum_{i=1}^{n} \left(H(Y_i) - H(Y_i \mid X_i) \right)$$

$$= 1 + nR \cdot P_e + \sum_{i=1}^{n} I(X_i; Y_i)$$

$$\leq 1 + nR \cdot P_e + \sum_{i=1}^{n} \left(\max_{p_X(x)} I(X;Y) \right)$$

$$= 1 + nR \cdot P_e + n \cdot \left(\max_{p_X(x)} I(X;Y) \right)$$

Since this inequality holds for all distribution of W, choose uniform W.

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$$H(W) \le \log \lceil 2^{nR} \rceil$$

$$\Rightarrow \underbrace{\log \lceil 2^{nR} \rceil}_{nR\ \le} \le \underline{1 + nR \cdot P_e + n\left(\max I(X;Y)\right)}$$

$$\Rightarrow R \le \frac{1}{n} + R \cdot P_{e,n} + \max_{p_X(x)} I(X;Y)$$

$$\Rightarrow R \le \max_{p_X(x)} I(X;Y) .$$