Lec. 19

2013년 4월 25일 목요일

오후 1:53

- **Review**

- Channel Coding Theorem for DMC.

  R is achievable iff $R \le \max_{p_X(x)} I(X;Y)$

  Achievability $\Rightarrow$

  Converse $\Leftarrow$

- Achievability

  (i) Find $\arg\max_{p_X(x)} I(X;Y)$

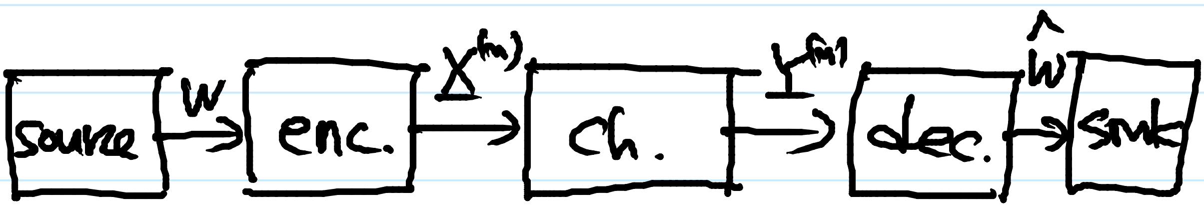
  (ii) Find the best code $\mathcal{C}_n'$ such that

  uniform r.v. W

  random encoder

  typical-set decoder

  $P_e(\mathcal{C}_n')$ is the minimum among all r. codes.



  (iii) Find $\tilde{\mathcal{C}}_n$ from $\mathcal{C}_n'$ by removing a half of the codewords.

$$2^{nR}/2 = 2^{nR-1} = 2^{n(R-\frac{1}{n})}$$

(iv) Choose $\varepsilon_k \to 0$

For each ε_k, we can find $\tilde{\mathcal{C}}_{nk}$.

$\therefore$ Then, $(\hat{\mathcal{C}}_{nk})_k$ has $R_k \to R$

$\varepsilon_k \to 0$.

- Converse

  - Fano's inequality
  - Data processing inequality
  - DO NOT USE the joint AEP!!
  - independence upper bound
    $I(X;Y) = H(Y) - H(Y|X)$
  - Chain rule
    $H(Y_1, Y_2, \cdots, Y_n | X)$
    $= \sum_{i=1}^{n} H(Y_i | Y_{i-1}, Y_{i-2}, \cdots, Y_1, X)$

$$H(W) = H(W|\hat{W}) + I(W;\hat{W})$$
$$= H(W|\underline{Y}^{(n)}) + I(W;\underline{Y}^{(n)}) \quad \text{D.P.I.}$$
$$= H(W|\underline{Y}^{(n)}) + I(\underline{X}^{(n)};\underline{Y}^{(n)})$$
$$\le 1 + nR\cdot P_{e,n} + I(\underline{X}^{(n)};\underline{Y}^{(n)}) \quad \text{Fano's inequality}$$
$$\le 1 + nR\cdot P_{e,n} + nC$$

$W \to \underline{X}^{(n)} \to \underline{Y}^{(n)}$ ...

A B C

$\underline{x}^{(m)} \to W \to \underline{x}^{(m)}$

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$$nR \le 1 + nR\cdot P_{e,n} + nC$$

$$\Rightarrow R \le \frac{1}{n} + R\cdot P_{e,n} + C$$

$$\Rightarrow R \le C \triangleq \max_{p_X(x)} I(X;Y)$$

- How to solve $\max_{p_X(x)} I(X;Y)$

- Example 1

$$X \quad \begin{matrix} 0 \longrightarrow 2 \\ 1 \longrightarrow 3 \end{matrix} \quad Y$$

deterministic, invertible } channel.

$\Downarrow$

error free

Experience

$\Downarrow$

Lesson. $C \overset{?}{=} \underbrace{1}$ [bits/channel use]

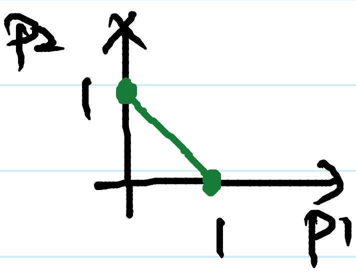
$$\begin{cases} \max_{x \in \Omega} f(x) \\ \max_{\underline{x} \in \Omega} f(\underline{x}) \end{cases}$$

$$\begin{bmatrix} p_X(0) \\ p_X(1) \end{bmatrix} = \underline{x}$$

$0 \le x_1 \le 1$

$0 \le x_2 \le 1$

$x_1 + x_2 = 1$



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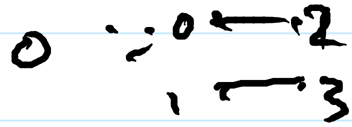
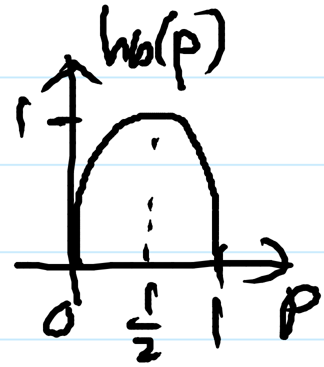
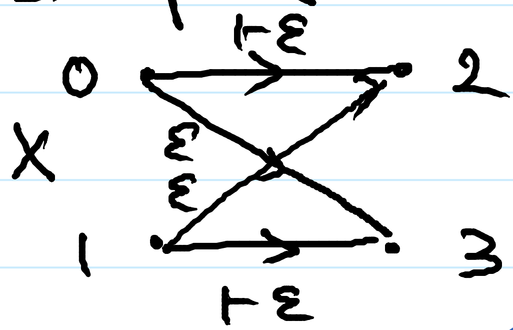
$$I(X;Y) = H(X) - H(X|Y)$$
$$= H(Y) - \cancel{H(Y|X)}$$
$$= H(X)$$
$$= h_b(x_1) \le 1$$
$$= \text{ holds iff } x_1 = \frac{1}{2}$$





$\therefore C = 1$ [bit/channel use]

- Example 2



Binary Symmetric Channel

$\varepsilon = 0 \Rightarrow C = 1$

$\varepsilon = 1 \Rightarrow C = 1$

$0 < \varepsilon < 1 \Rightarrow C \le 1$

$\varepsilon = 1/2 \Rightarrow C \overset{?}{=} 0$

what we know

intuition

symmetry

$$\therefore C = 1 - h_b(\varepsilon) \text{ [bits/ch. use]}$$

sol) $\max_{p_X(x)} I(X;Y)$

$$\equiv \max_{x \in \mathbb{R}^2} I(X;Y)$$

s.t. $0 \le x_1, x_2 \le 1$, $x_1 + x_2 = 1$.

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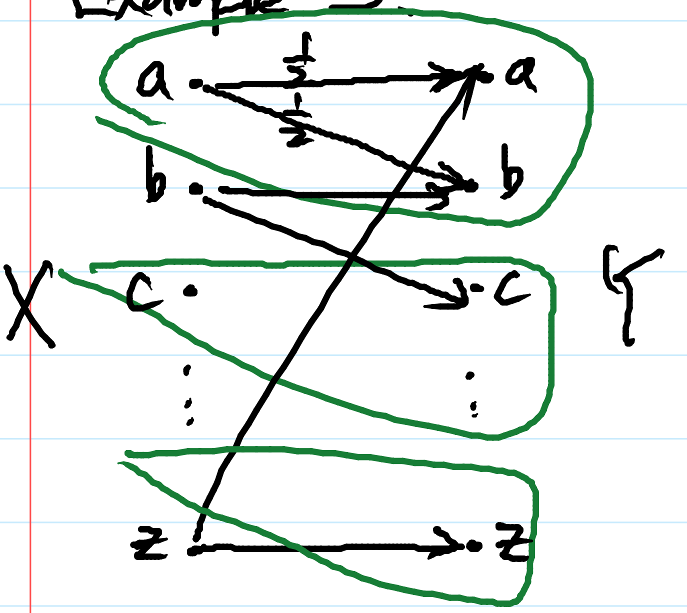
- Example 3



A~Z : 26

ㄱ~ㅎ : 14

ㅏ~ㅣ : 10

· ㅿ ㅸ ㆆ : 4

Noisy Typewriter Ch.

(can be viewed as a generalization BSC w/ $\varepsilon = 1/2$)

$$P_{Y|X} \triangleq \begin{bmatrix} P_{Y|X}(y_1|x_1) & P_{Y|X}(y_2|x_1) & \cdots \\ & & \\ & & \end{bmatrix}$$

BSC $\begin{bmatrix} 1-\varepsilon & \varepsilon \\ \varepsilon & 1-\varepsilon \end{bmatrix}$

NTC $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & & \\ & \frac{1}{2} & \frac{1}{2} & \\ & & & \ddots \end{bmatrix}$

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- Symmetric Channel

$P_{Y|X}$ matrix has each row obtained by permuting the 1st row $\underline{r}$.

$$\max_{p_X(x)} I(X;Y)$$

$$= \max_{p_X(x)} H(Y) - \underbrace{H(Y|X)}_{\sum_x H(Y|X=x)\Pr(X=x)}$$

$$= H(\underline{r})$$

$$= \sum_i r_i \log \frac{1}{r_i}$$

$$= \max_{p_X(x)} H(Y) - H(\underline{r})$$

$$\le \log_2 |\mathcal{Y}| - H(\underline{r})$$

rex) BSC $\Rightarrow C \le \log_2 |\{2,3\}| - h_b(\varepsilon)$

$= 1 - h_b(\varepsilon)$

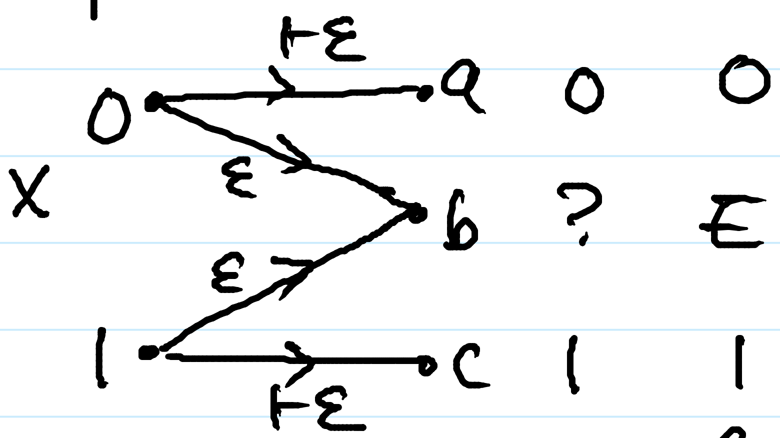
if $p_X(0) = p_X(1) = 1/2$.

NTC $\Rightarrow C \le \log_2 26 - h_b(\frac{1}{2})$

$= \log_2 26 - 1 = \log_2 13$

if $p_X(a) = p_X(b) = \cdots = p_X(z) = 1/26$.

- Example 4



$\underbrace{\text{Binary}}_{\text{input}}$ Erasure Channel (BEC)

$$P_{Y|X} = \begin{bmatrix} 1-\varepsilon & \varepsilon & 0 \\ 0 & \varepsilon & 1-\varepsilon \end{bmatrix}$$

$$C \le \log_2 3 - h_b(\varepsilon)$$

☹.

$$C_{BEC} = 1 - \varepsilon \ [\text{bits/ch.use}]$$