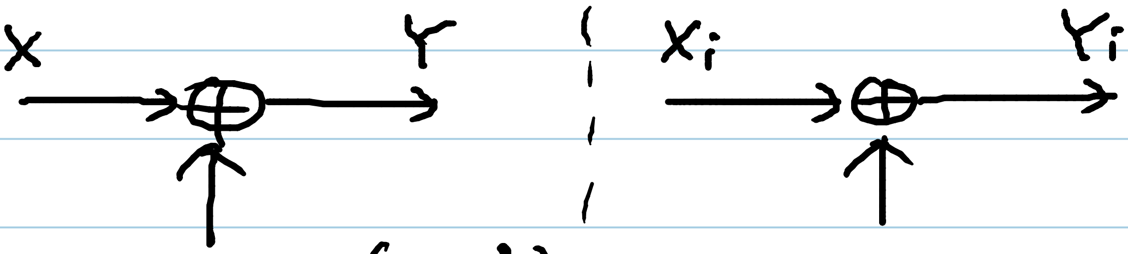
Lec. 25

2013년 3월 5일 화요일

오후 2:33

memoryless extension

Review



$Z \sim N(0, \sigma_Z^2)$ | $Z_i \sim N(0, \sigma_Z^2)$

Z_i's are i.i.d.

$$C_I \triangleq \max_{f_X(x)} I(X;Y)$$

s.t. $E\{X^2\} \leq P$

$C_o \triangleq \sup R$

Def. A rate R is achievable if there exists a sequence of (M_k, n_k) codes $(\mathcal{C}_k)_{k=1}^{\infty}$ such that

(i) $\|\underline{x}_i^{(n_k)}\|^2 \leq n_k P, \ \forall k, \forall 1 \leq i \leq M_k$.

(ii) $R_k \triangleq \dfrac{\log_2 M_k}{n_k} \to R$ as $k \to \infty$,

(iii) $\lambda^{(n_k)} \triangleq \max_{1 \leq i \leq M_k} \Pr(\hat{W} \neq W \mid W = m_i, \mathcal{C}_k)$

$\longrightarrow 0$ as $k \to \infty$

- NCCT for DT AWGN channel

$$C_o = C_I$$

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- NCCT for CMC

$C_I \triangleq \sup_{f_X(x)} I(X;Y)$ s.t. $E\{b(X)\} \le \beta$

β-admissibility constraint

$C_I(\beta)$ capacity-cost function

$C_O \triangleq \sup R$

A rate R is achievable if there exists a seq. of (M_k, n_k) code $(\mathcal{C}_k)_{k=1}^{\infty}$ s.t.

(i) $\frac{1}{M_k}\sum_{j=1}^{m_k} b(x_{ij}^{(m_k)}) \le \beta$, $\forall i,j,k$.

(ii) $R_k \triangleq \frac{\log_2 M_k}{n_k} \to R$ as $k \to \infty$, and

(iii) $\lambda^{(M_k)} \triangleq \max_{1 \le i \le M_k} \Pr(\hat{W} \ne W \mid W = m_i; \mathcal{C}_k)$

$\to 0$ as $k \to \infty$.

$C_I = C_O$.

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- C_I of DT AWGN

$$C_I = \frac{1}{2}\log_2\left(1+\frac{P}{\sigma_Z^2}\right) \text{ [bits/ch.use]}$$

$$X \sim N(0, P)$$

- C_I is achievable.

(i) Choose $f_X(x) = \frac{1}{\sqrt{2\pi(P-\varepsilon')}}\exp\left(-\frac{x^2}{2(P-\varepsilon')}\right)$

$\varepsilon' > 0$

(ii) Generate random $(\lceil 2^{nR}\rceil, n)$ codes $\mathcal{C}_n$

(iii) The codebook is shared by encoder & decoder

(iv) The decoder employs a modified typical set decoding rule: Given $\{\underline{Y}^{(n)} = \underline{y}^{(n)}\}$,

$\hat{w} = m_i$ if $\|\underline{x}_i^{(n)}\|^2 \le nP$,

$(\underline{x}_i^{(n)}, \underline{y}^{(n)}) \in A_\varepsilon^{(n)}$, and

$(\underline{x}_j^{(n)}, \underline{y}^{(n)}) \notin A_\varepsilon^{(n)}$, $\forall j \ne i$.

declare error otherwise.

- $\Pr(\hat{W} \ne m_i \mid W = m_i)$ ← averaged over all random codes.

$$\le \varepsilon + \varepsilon + 2^{nR}\, 2^{-n(I(X;Y)-3\varepsilon)}$$

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$$2^{nR}\, 2^{-n(I(X;Y)-3\varepsilon)}$$

$$= 2^{-n(I(X;Y)-R-3\varepsilon)}$$

We want

$I(X;Y)-3\varepsilon > R$

$= \frac{1}{2}\log\left(1+\frac{P-\varepsilon'}{\sigma_Z^2}\right)$

-3ε

To construct a seq.

(i) We choose $\varepsilon_k' > 0$ & $\varepsilon_k > 0$

(ii) We find n_k s.t. $\Pr(\hat{W} \neq m_i \mid W = m_i) < 3\varepsilon_k$

(i) Consider uniform message distribution

(ii) Choose $\varepsilon_k' > 0$, & $\varepsilon_k > 0$

(iii) We find n_k s.t. $\Pr(\hat{W} \neq m_i \mid W = m_i) < 3\varepsilon_k$

(iv) We find $\mathcal{C}_{n_k}$ s.t. $P_e(\mathcal{C}_{n_k})$ is the minimum of all $(\lceil 2^{n_k R} \rceil, n_k)$ random codes.

(v) Throw away around the half of the codewords to find $\hat{\mathcal{C}}_{n_k}$.

$R < C_I$ is achievable.

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discrete

$W \to \underline{X}^{(n)} \to \underline{Y}^{(n)}$ $\quad I(W, \underline{Y}^{(n)}) \le I(\underline{X}^{(n)}, \underline{Y}^{(n)})$

- Converse. $<$ Data Processing Ineq. / Fano's ... $H(W|\underline{Y}^{(n)})$ conditioning reduces the entropy.

To show:

"If there exists a seq. of $(\lceil 2^{n_k R} \rceil, n_k)$ codes $(\mathcal{C}_k)_{k=1}^{\infty}$ such that

~~random~~

(i) $\|\underline{x}_i^{(n_k)}\|^2 \le nP, \ \forall i, \forall k$ &

(ii) $\lambda^{(n_k)} \to 0$ as $k \to \infty$

then $R \le C_I$."

Fano's inequality derivation:

- $$\begin{aligned} H(W|\underline{Y}^{(n)}) &= H(W|\underline{Y}^{(n)}) + \overbrace{H(E|W,\underline{Y}^{(n)})}^{0} \\ &= H(W,E|\underline{Y}^{(n)}) \\ &= H(E|\underline{Y}^{(n)}) + H(W|E,\underline{Y}^{(n)}) \\ &\le 1 + nR \cdot P_e \end{aligned}$$

$W \to \underline{X}^{(n)} \to \underline{Y}^{(n)}$

- $$\begin{aligned} H(W) &\stackrel{\Delta}{=} H(W|\underline{Y}^{(n)}) + I(W;\underline{Y}^{(n)}) \\ &\le 1 + nR \cdot P_e + I(\underline{X}^{(n)};\underline{Y}^{(n)}) \\ &= 1 + nR \cdot P_e + \cancel{H(\underline{X}^{(n)}) - H(\underline{X}^{(n)}|\underline{Y}^{(n)})} \\ &\qquad h(\underline{Y}^{(n)}) - h(\underline{Y}^{(n)}|\underline{X}^{(n)}) \end{aligned}$$

$$\leq \; 1 + nR \cdot P_e + \sum_{i=1}^{n} h(Y_i) - \sum h(Y_i | X_i)$$

$$\le 1 + nR \cdot P_e + \sum_{i=1}^{n} \frac{1}{2} \log_2 \left(1 + \frac{P_i}{\sigma_Z^2}\right) \frac{n}{n}$$

$$\le 1 + nR \cdot P_e + n \cdot \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma_Z^2}\right)$$

Jensen's inequality $\sum P_i \le nP$

A Property Concave function

f is concave iff

$$f(\lambda x + (1-\lambda) y) \ge \lambda f(x) + (1-\lambda) f(y)$$

$$\Leftrightarrow f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \ge \sum_{i=1}^{n} \lambda_i f(x_i)$$

$$f(E(X)) \ge E\{f(X)\}$$

Jensen's inequality

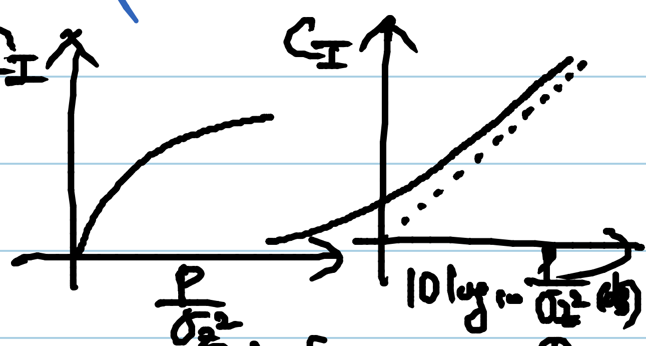
- Choose uniform W.

$$\log 2^{nR} \le H(W) \le$$

$$\cancel{n}R \le \frac{1}{n} + \cancel{n}R \cdot P_e^{(n)} + \cancel{n} \cdot \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma_Z^2}\right)$$

Let $n \to \infty$

$$R \le C_I.$$



$$\therefore \; C_O = C_I = \frac{1}{2} \log_2 \left(1 + \frac{P}{\sigma_Z^2}\right)$$

[bits/ch. use]

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$$= 1 + nR \cdot P_e + \sum_{i=1}^{n} I(X_i ; Y_i)$$

$$E\{X_i\} = P_i$$

$$\sum_{i=1}^{n} P_i \le nP$$

$$\le 1 + nR \cdot P_e +$$