Lec. 27

2013년 5월 17일 금요일

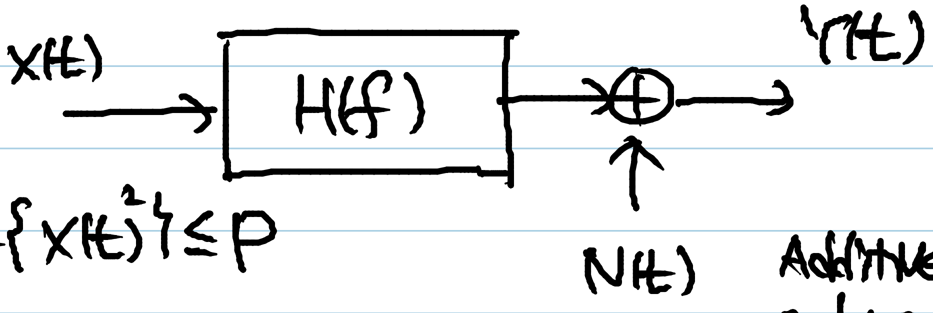
오전 8:56

· Review

extension

generalization

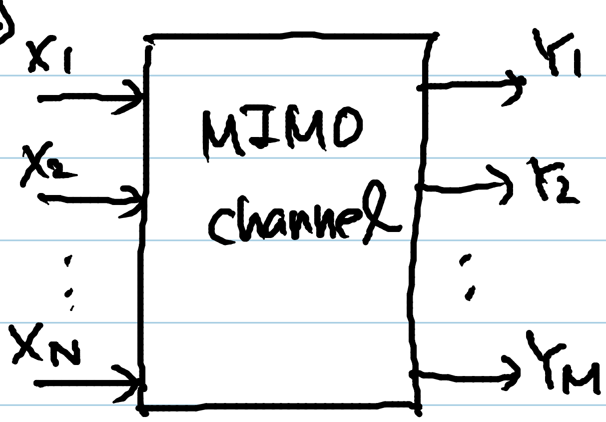
DT AWGN channel

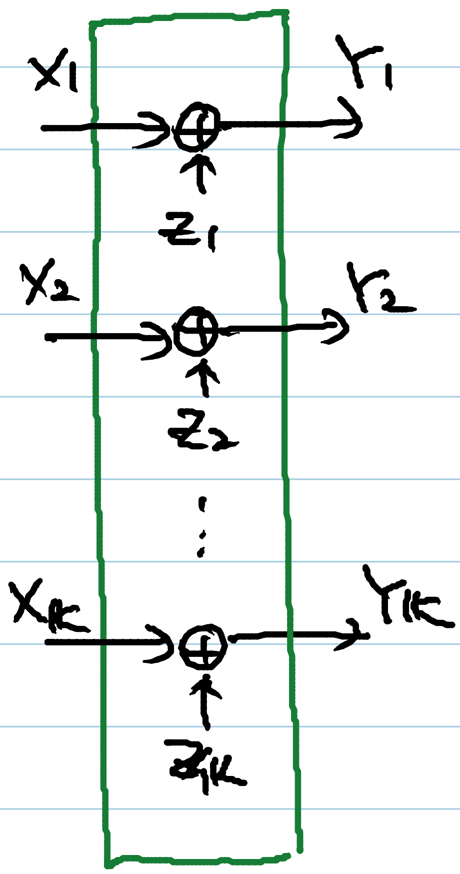


$E\{X(t)^2\} \le P$

Additive colored Gaussian random process

simplify





Z_i's are independent

$Z_i \sim N(0, \sigma_i^2)$

- Informational Channel Capacity of
  - parallel Gaussian channel
  - Gaussian product channel

$$C_I \triangleq \max_{f_X(x)} I(X;Y) \qquad f_{Y|X}(y|x)$$

$$\text{s.t. } E\{\|X\|^2\} \le P$$

$$= \prod_{i=1}^{K} f_{Y_i|X_i}(y_i|x_i)$$

$$= \prod_{i=1}^{K} \frac{1}{\sqrt{2\pi\sigma_i^2}} e^{-\frac{(y_i-x_i)^2}{2\sigma_i^2}}$$

backward compatibility

$$I(X;Y) = h(Y) - h(Y|X) \qquad \int_x h(Y|X=x) f_X(x)\,dx$$

$$\le \sum_{i=1}^{K} h(Y_i) - h(Z) \qquad Y = x + Z$$

$$= \sum_{i=1}^{K} \{h(Y_i) - h(Z_i)\} \qquad h(Y_i|X_i) = h(Z_i)$$

$$\le \sum_{i=1}^{K} \frac{1}{2}\log\left(1+\frac{P_i}{\sigma_i^2}\right)$$

$P_i \triangleq E\{X_i^2\}$

$X_i \sim N(0, P_i)$

$\sum_{i=1}^{K} P_i \le P$

$$\Rightarrow \quad C_I = \max_{P} \sum_{i=1}^{K} \frac{1}{2}\log\left(1+\frac{P_i}{\sigma_i^2}\right)$$

$$\text{s.t. } 0 \le P_i, \ \forall i$$

$$\sum_{i=1}^{K} P_i \le P$$

ex/ extre-mization

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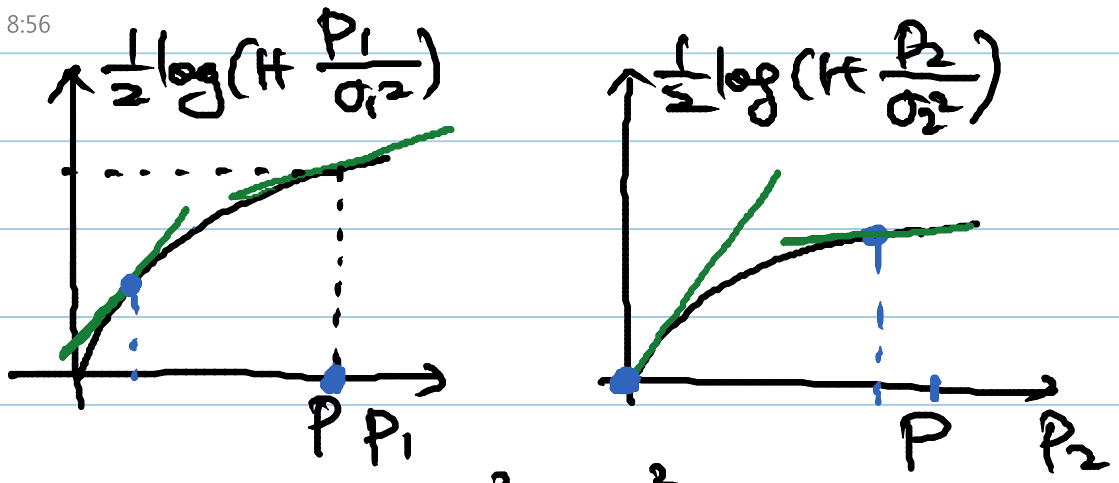

$$\sigma_1^2 < \sigma_2^2$$

$$(P_1, P_2) = (P, 0)$$
$$(P-\varepsilon, \varepsilon)$$

Lesson

→ If (P_1^*, P_2^*) is an optimal solution,

then $\frac{d}{dP_1}\frac{1}{2}\log(1+\frac{P_1}{\sigma_1^2})\Big|_{P_1^*}$

$$= \frac{d}{dP_2}\frac{1}{2}\log(1+\frac{P_2}{\sigma_2^2})\Big|_{P_2^*}$$

$\log_2 x$
$= \log_2 e \cdot \ln x$

Generalization

→ Set λ be the common slope.

$$\cancel{\log_2 e}\ \cancel{\frac{1}{2}}\ \frac{\frac{1}{\sigma_i^2}}{1+\frac{P_i}{\sigma_i^2}} = \tilde{\lambda},\ \forall i$$

$$\Rightarrow \frac{1}{P_i+\sigma_i^2} = \tilde{\lambda}$$

$$\Rightarrow P_i = \nu - \sigma_i^2,\ \forall i$$

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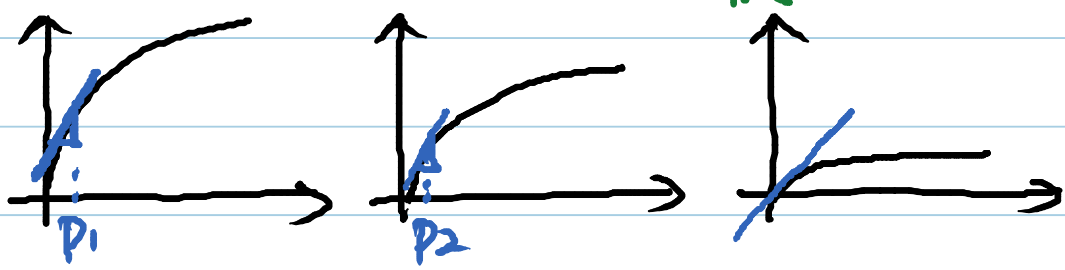
- Caution

Since $P_i = \nu - \sigma_i^2, \ \forall i$

$$\sum_{i=1}^{K} P_i = \sum_{i=1}^{K} (\nu - \sigma_i^2) = P \quad \text{implies}$$

$$\cancel{K}\nu = \frac{P + \sum_{i=1}^{K} \sigma_i^2}{K}$$



$P_1 + P_2 = P$

$$P_i = [\nu - \sigma_i^2]^+, \ \forall i$$

$$[x]^+ = \max(0, x) = \frac{x + |x|}{2}$$

$\therefore$ Find the unique solution $\nu^* > 0$ to

$$\sum_{i=1}^{K} [\nu^* - \sigma_i^2]^+ = P$$

$\Rightarrow$ Then, $P_i^* = [\nu^* - \sigma_i^2]^+$.

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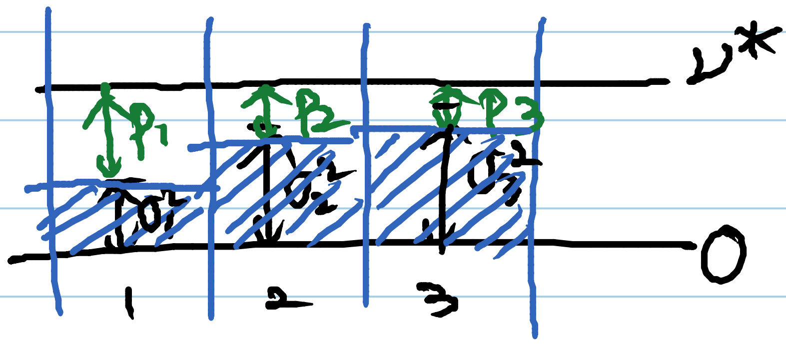
Water-pouring algorithm
- Water-filling procedure
argument

$\underline{y} = H\underline{x} + \underline{n}$

$$P_i = [\nu^* - \sigma_i^2]^+$$



(cf) Lagrange multiplier
KKT condition
convex optimization
⋮

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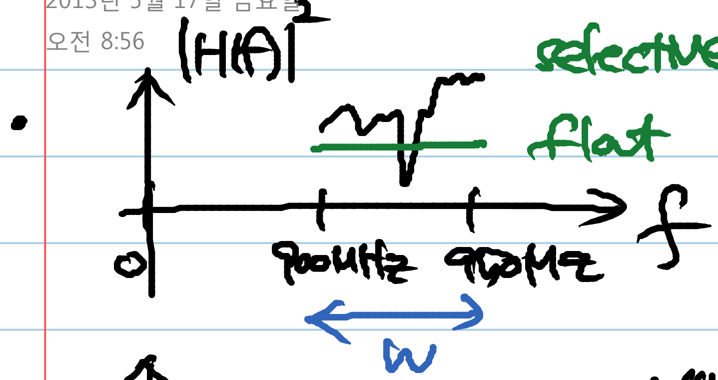
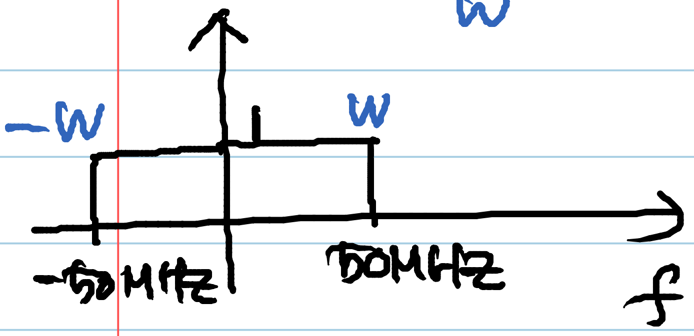
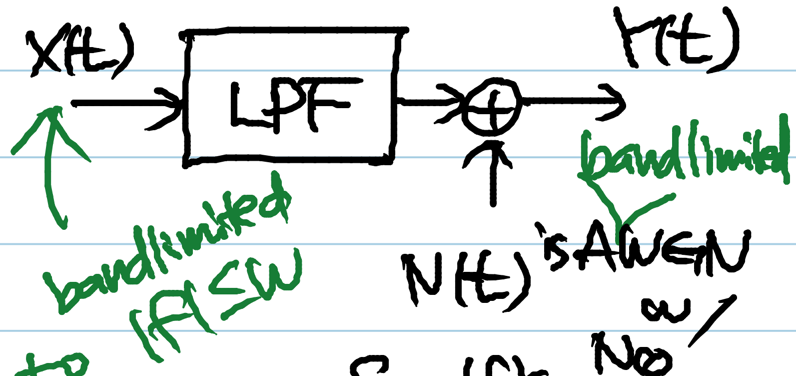
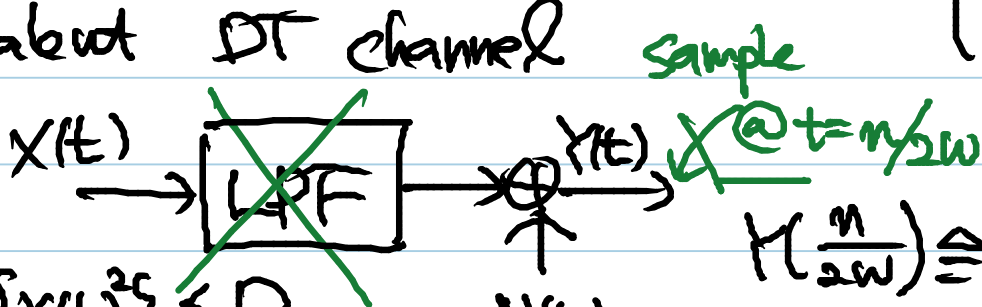






$$S_{NN}(f) = \begin{cases} \frac{N_0}{2}, & |f| \le W \\ 0, & \text{elsewhere} \end{cases}$$

- Equivalent DT channel



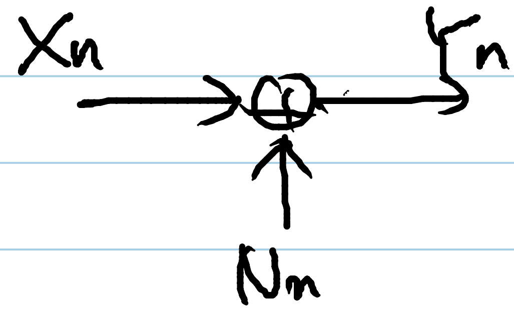
$E\{X(t)^2\} \le P$

$$Y\left(\frac{n}{2W}\right) \triangleq Y_n$$

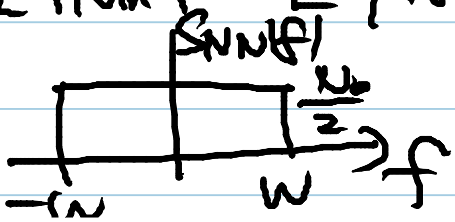
$$= X\left(\frac{n}{2W}\right) + N\left(\frac{n}{2W}\right)$$

$$\triangleq X_n + N_n$$

flat PSD

$\equiv$ 

(i) $E\{N_n^2\} = E\left\{N\left(\frac{n}{2W}\right)^2\right\} = E\{N(t)^2\}$



$= N_0 W$

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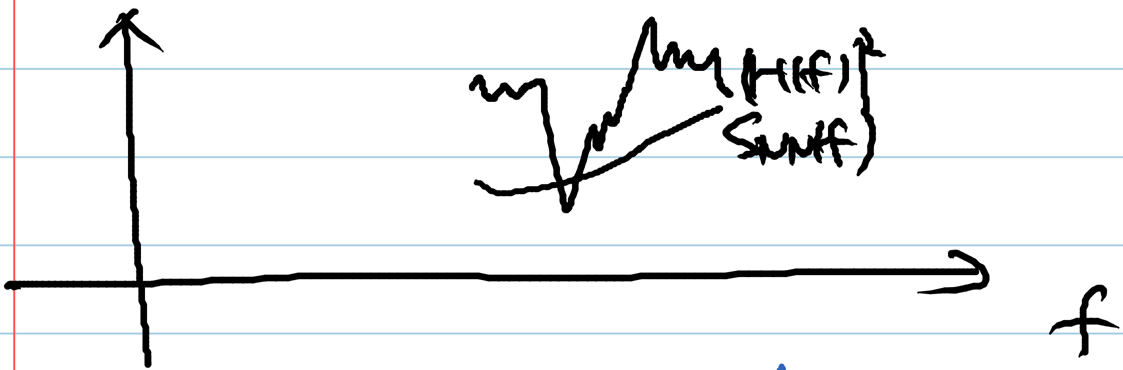
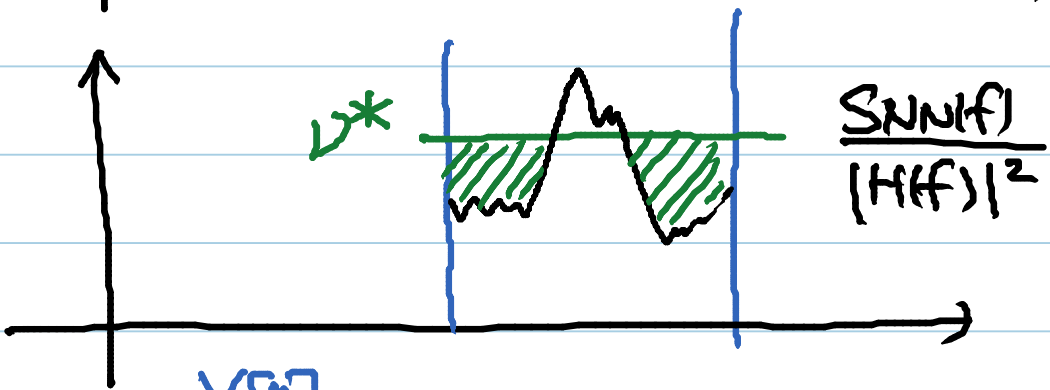
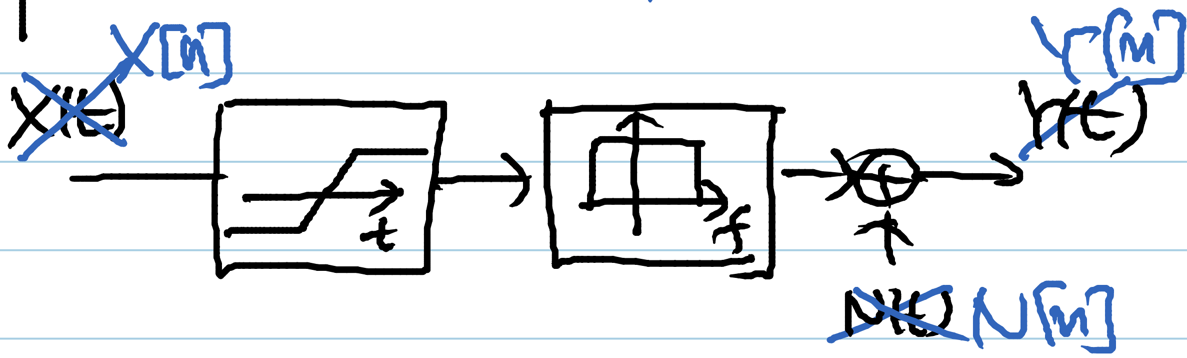
오전 10:28







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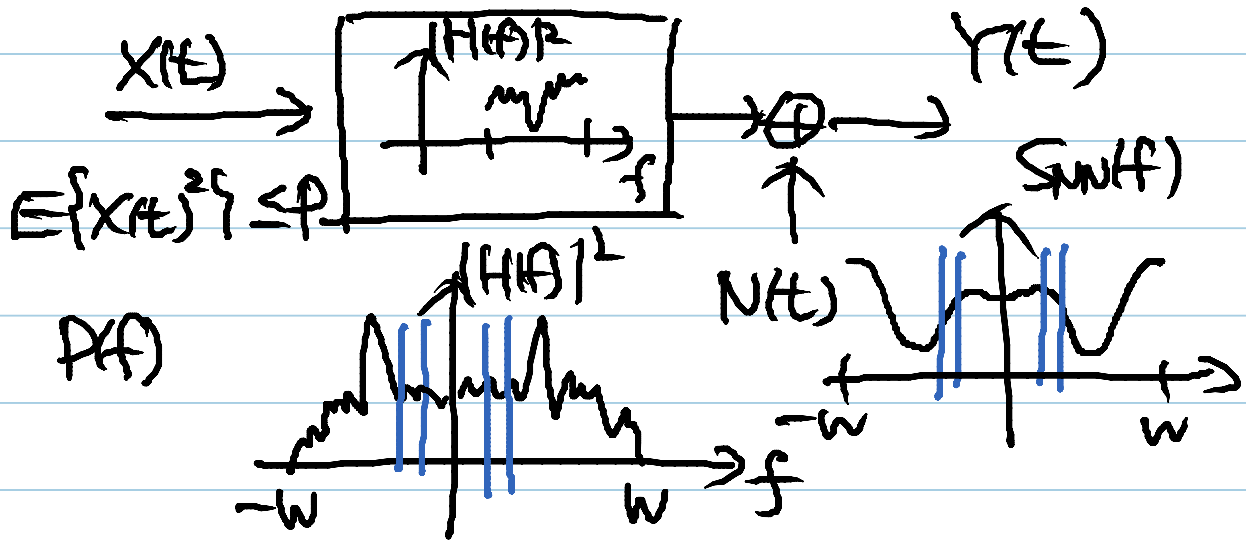
오전 10:16

- Capacity of Frequency-Selective AWGN bandlimited channel



Newton's Lesson

$$\max_{P(f)} \int_{-W}^{W} (df) \log\left(1+\frac{P(f)}{\frac{S_{NN}(f)}{|H(f)|^2}}\right) df$$

$$\int_{-W}^{W} P(f)\, df \le P$$

$$0 \le P(f), \quad \forall |f| \le W$$

$\Rightarrow$ Find the unique solution ν^* to

$$\int_{-W}^{W} \left[\nu^* - \frac{S_{NN}(f)}{|H(f)|^2}\right]^+ df = P$$

Then, $P(f) = \left[\nu^* - \frac{S_{NN}(f)}{|H(f)|^2}\right]^+, \ \forall |f| \le W$

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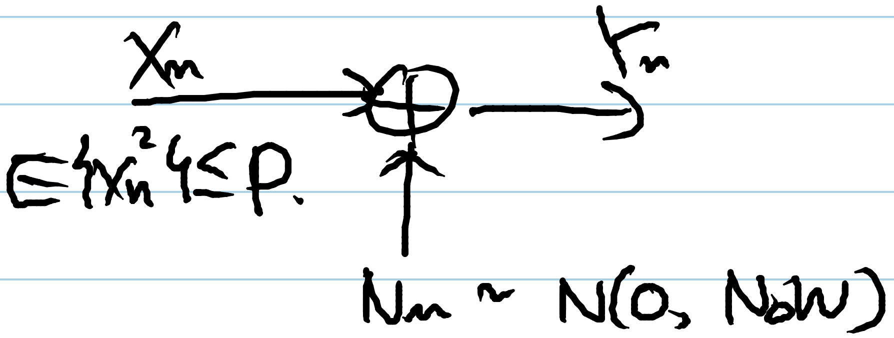
$$E\{X(t)^2\} \le P, \ \forall t$$

$$E\{X_n^2\} = E\{X(\tfrac{n}{2W})^2\} \le P$$



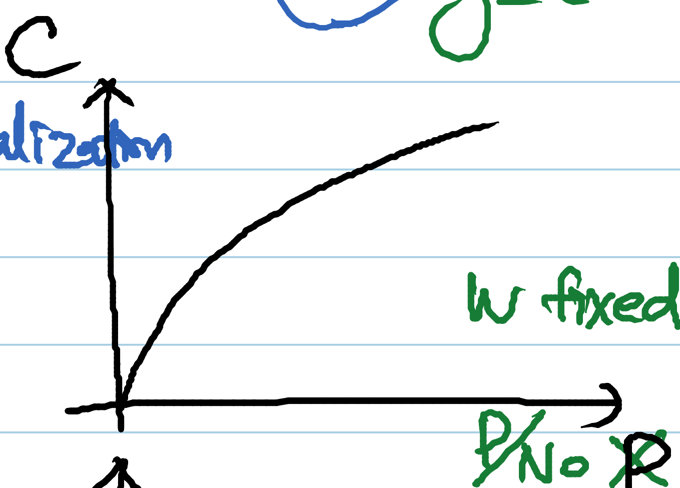
$$C_{equiv.DT} = \frac{1}{2}\log_2\left(1+\frac{P}{N_0W}\right) \ [\text{bits/ch.use}]$$

$$C_{CT} = C_{equiv.DT} \times 2W$$

[bits/ch.use] × [ch.use/sec]

$$= W\log_2\left(1+\frac{P}{N_0W}\right) \ [\text{bps}]$$

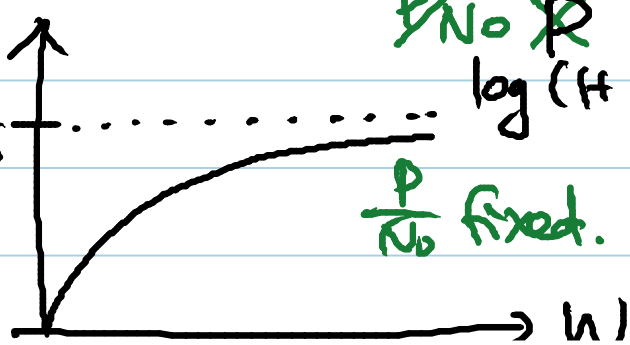
· Visualization



W

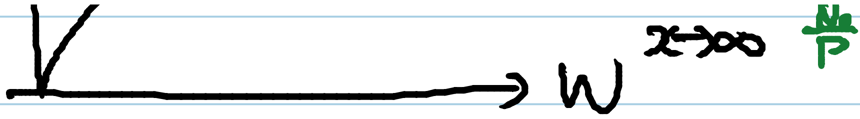
P

$N_0/2$



$$\log\left(1+\frac{P}{N_0W}\right)^W = (\log_2 e)\frac{P}{N_0}.$$

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x\frac{P}{N_0}} = e^{\frac{P}{N_0}}, \quad x = \frac{N_0W}{P}$$


W
x→∞