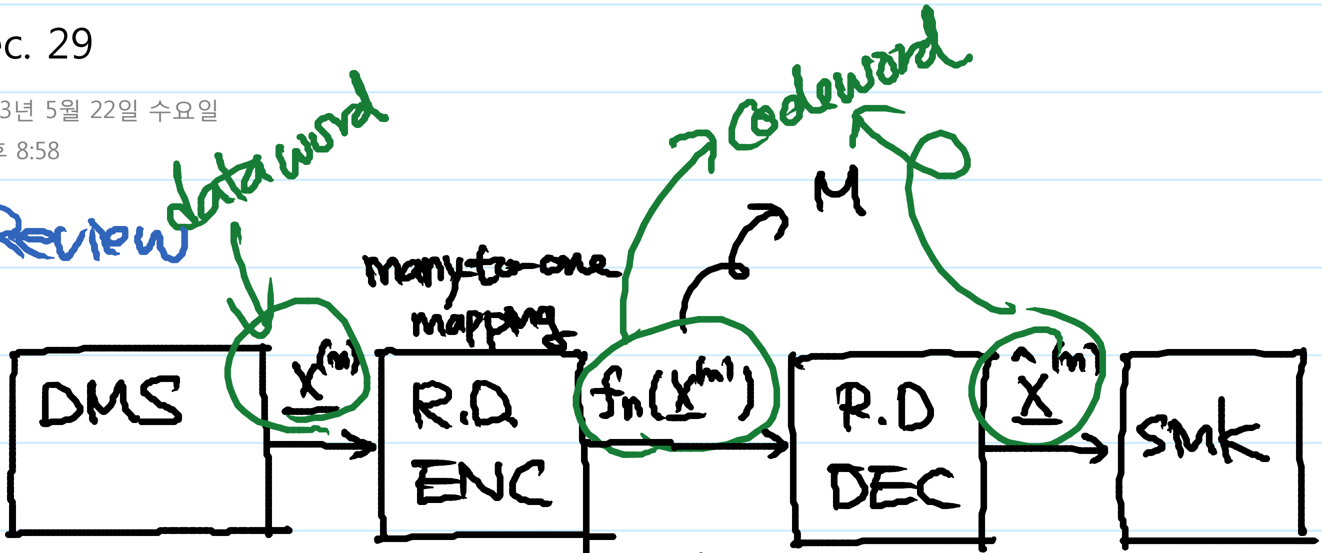
Lec. 29

2013년 5월 22일 수요일

오후 8:58

- Review



$\mathcal{X}$: source alphabet

$E\{d(\underline{x}^{(n)}, \hat{\underline{x}}^{(n)})\} \le D$

$$\min \frac{\log_2 M}{n} \quad [\text{bits/source output}]$$

$$\text{s.t. } E\{d(\underline{x}^{(n)}, \hat{\underline{x}}^{(n)})\} \le D$$

- Def.

A rate distortion pair (R, D) is achievable if there exists $(2^{n_k R}, n_k)$ rate distortion codes such that

$$\lim_{k \to \infty} E\{d(\underline{x}^{(n_k)}, \hat{\underline{x}}^{(n_k)})\} \le D.$$

an (M, n) rate distortion code

compression 後

# of source outputs

- To show, if $R < R_I(D) = \min\limits_{\substack{p_{\hat{X}|X} \\ \text{s.t. } E\{d(X,\hat{X})\} \le D}} I(X;\hat{X})$

then (R, D) is achievable.

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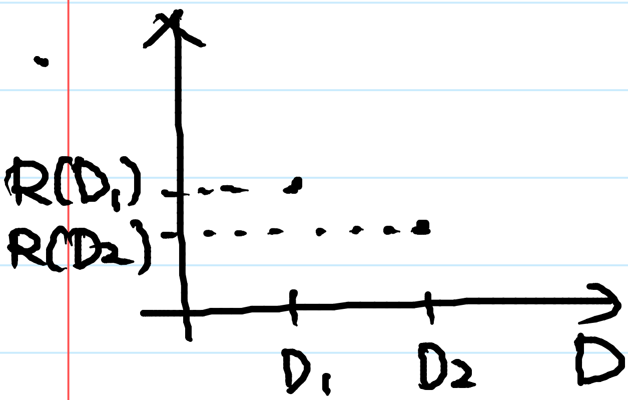
· To show, if (R, D) is achievable,

then $R \geq R_I(D)$.

· 

$$R(D_1) = \inf_{(R, D_1) \text{ is achievable.}} R$$

$$R(D_2) = \inf_{(R, D_2) \text{ is achievable}} R$$

Q. $R(D)$

· Lemma.

$R_I(D)$ as a function of $D \geq 0$ is a monotonically decreasing convex function

(proof)

Suppose that $D_1 \leq D_2$.

$$R_I(D_1) = \min_{p_{\hat{X}|X}} I(X;\hat{X}) \quad \text{s.t. } E\{d(X,\hat{X})\} \leq D_1 \qquad \Omega_1$$

$$\geq R_I(D_2) = \min_{p_{\hat{X}|X}} I(X;\hat{X}) \quad \text{s.t. } E\{d(X,\hat{X})\} \leq D_2 \qquad \Omega_2$$

$$\Omega_1 \subset \Omega_2$$

To show If $\lambda \in [0,1]$

$D_\lambda \triangleq \lambda D_1 + (1-\lambda) D_2$

then $R_I(D_\lambda) \le \lambda R_I(D_1) + (1-\lambda) R_I(D_2)$

• A detour

$$R_I(D) = \min_{p_{\hat X|X}} I(X;\hat X)$$

$$\text{s.t. } E\{d(X,\hat X)\} \le D$$

To show, if $\lambda \in [0,1]$

$p_1(\hat x|x)\,p(x) = p_1(x,\hat x)$ s.t. $E_1\{d(X,\hat X)\} \le D$

$p_2(\hat x|x)\,p(x) = p_2(x,\hat x)$ s.t. $E_2\{d(X,\hat X)\} \le D$

then

$$p_\lambda(\hat x|x) = \lambda p_1(\hat x|x) + (1-\lambda) p_2(\hat x|x)$$

$$\Leftrightarrow p_\lambda(x,\hat x) = \lambda p_1(x,\hat x) + (1-\lambda) p_2(x,\hat x)$$

$$E_\lambda\{d(X,\hat X)\} = E_1\{d(X,\hat X)\}P_1 + E_2\{d(X,\hat X)\}P_2$$
$$\le \lambda D + (1-\lambda) D = D.$$

∴ The constraint set is a cvx set.

To show, $I_\lambda(X;\hat X) \le \lambda I_1(X;\hat X) +$

$$(1-\lambda) I_2(X;\hat{X})$$

$$I_x(X;\hat{X}) = E\left\{\log \frac{P_x(X,\hat{X})}{P(X)P_x(\hat{X})}\right\} \quad \cdots\cdots (*)$$

$$I_1(X;\hat{X}) = E\left\{\log \frac{P_1(X,\hat{X})}{P(X)P_1(\hat{X})}\right\} \quad \cdots\cdots (**)$$

$$I_2(X;\hat{X}) = E\left\{\log \frac{P_2(X,\hat{X})}{P(X)P_2(\hat{X})}\right\} \quad \cdots\cdots (***)$$

- Log-sum inequality

$a_i > 0$

$b_i > 0$

$$a_1 \log \frac{a_1}{b_1} + a_2 \log \frac{a_2}{b_2} \geq (a_1+a_2) \log \frac{(a_1+a_2)}{(b_1+b_2)}$$

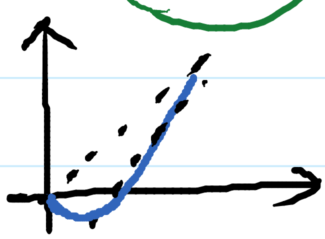
(sol) " f is cvx.

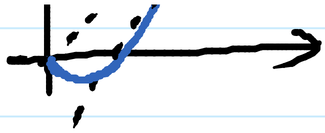
$\Rightarrow E[f(X)] \geq f(E[X])$ "

$$\sum_i p_i f(x_i) \geq f\left(\sum p_i x_i\right)$$

Choose $p_i = \dfrac{b_i}{\sum_{j=1}^{n} b_j}$, $x_i = \dfrac{a_i}{b_i}$

$f(x) = x \log x$



$f(x) = x\log x$ 

$$f(E(X)) = f\left(\sum_i p_i x_i\right)$$

$$\sum_i \frac{a_i}{\cancel{\sum_j b_j}} \log \frac{a_i}{b_i} \geq \frac{\sum a_i}{\cancel{\sum_j b_j}} \log \frac{\sum a_i}{\sum_j b_j}$$

$$\Rightarrow \sum_i a_i \log \frac{a_i}{b_i} \geq \left(\sum_i a_i\right) \log \frac{(\sum a_i)}{(\sum b_j)}$$

- Back to the convexity of $R_I(D)$

$D_1 < D_2$, λ

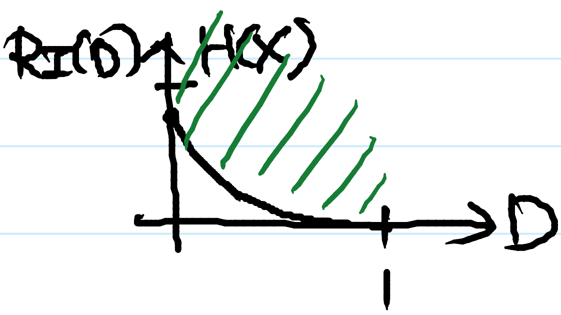
$$\lambda R_I(D_1) + (1-\lambda) R_I(D_2) \geq R_I(\lambda D_1 + (1-\lambda) D_2)$$

$$\lambda I_1(X;\hat{X}) + (1-\lambda) I_2(X;\hat{X}) \geq I_\lambda(X;\hat{X})$$

$$E_{\hat{X}}[d(X,\hat{X})] \leq \lambda D_1 + (1-\lambda) D_2$$

$$\geq R_I(\lambda D_1 + (1-\lambda) D_2).$$

$\therefore$ $R_I(D)$ is a cvx function of D



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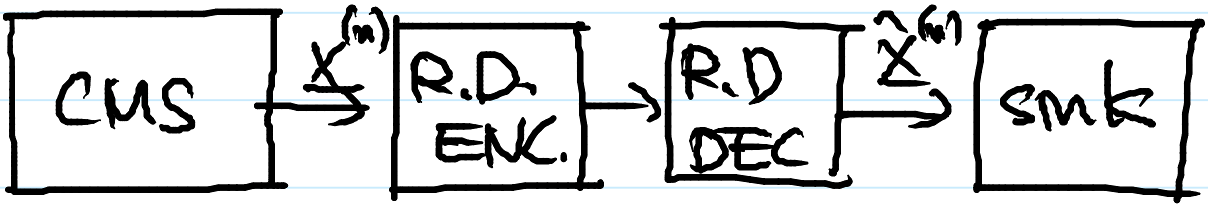
- Example.



$X \sim N(0,\sigma^2)$

Q. Find $R_I(D)$ when $d(x,\hat{x}) = (x-\hat{x})^2$, $D < \sigma^2$.

$$\min_{p_{\hat{x}|x}} I(X;\hat{X}) \quad \text{s.t. } E\{d(X,\hat{X})\} \le D$$

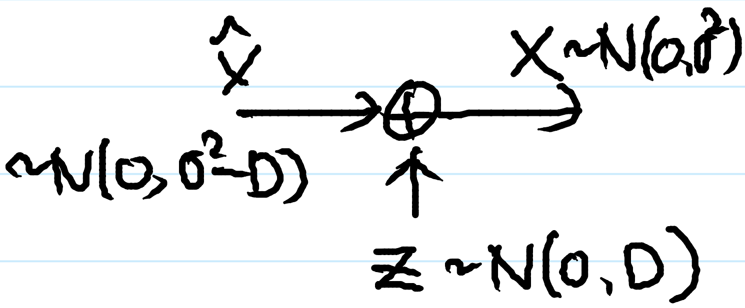
$P_{\hat{x}|x}\ P_x$

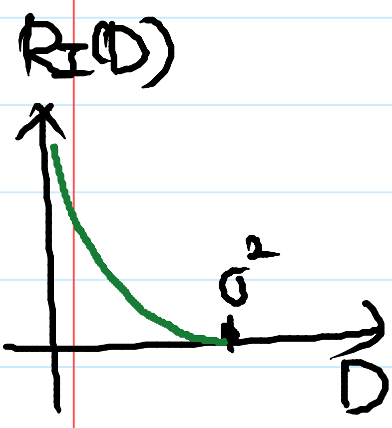
$f_{\hat{x}|x}\ f_x$

$$I(X;\hat{X}) = h(X) - h(X|\hat{X})$$

$$= \cancel{h(\hat{X}) - h(\hat{X}|X)}$$

$$= \frac{1}{2}\log(2\pi e\sigma^2) - \int_{\hat{x}} h(X|\hat{X}=\hat{x}) f(\hat{X}=\hat{x})$$





$$\ge \frac{1}{2}\log(2\pi e\sigma^2) - \frac{1}{2}\log(2\pi e D)$$

$$= \frac{1}{2}\log\frac{\sigma^2}{D}, \quad D < \sigma^2$$

- To show $\boxed{\text{the rate-distortion theorem}}$ in the next meeting.

  - ☆ achievability : If $R > R_I(D)$, then (R, D) is achievable
  - ☆ converse : If (R, D) is achievable, then $R \geq R_I(D)$.

- (R, D) is achievable.
- $R_I(D)$ is a monotone decreasing convex function of D.

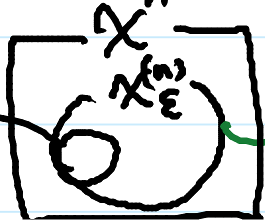
  ($\because$ $I(X;\hat{X})$ is a cvx function of $P_{\hat{X}|X}$).

  ($R_I(D) = \min\limits_{P_{\hat{X}|X}} I(X;\hat{X})$ is obtained from

  s.t. $E\{d(X,\hat{X})\} \leq D$

  the sphere covering argument.)

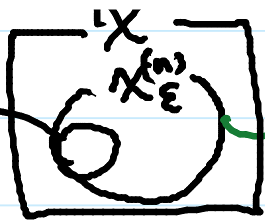
- Example. $\boxed{\begin{matrix}\text{Gaussian} \\ \omega/\sigma^2\end{matrix}} \rightarrow \frac{1}{2}\log\frac{\sigma^2}{D}$ [bits/s.o]



$\text{Vol} \sim 2^{-n\frac{1}{2}\log 2\pi e D}$

$\rightarrow \text{Vol}(X_\varepsilon^{(n)}) \cong 2^{-nh(X)} = 2^{-n\cdot\frac{1}{2}\log 2\pi e\sigma^2}$

$\mathrm{Vol} \simeq 2^{-n\frac{1}{2}\log\ldots}$



$$\mathrm{Vol}(X_\varepsilon^{(n)}) \cong 2^{-nh(X)} = 2^{-n\cdot\frac{1}{2}\log 2\pi e\sigma^2}$$