Lec. 31

2013년 5월 29일 수요일
오후 9:03

- Overview

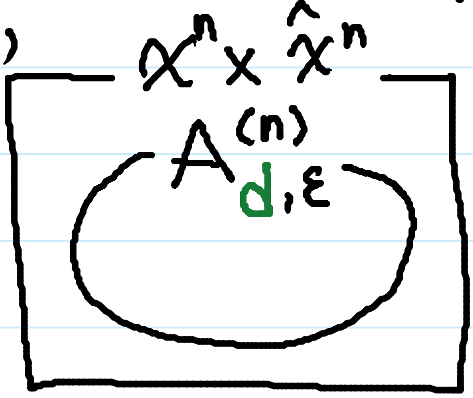
  - data compression
    - Zero-error variable-length
    - arbitrarily small-error block
    - lossy block
  - data transmission

- Rate-distortion theorem

$$R > R_I(D) \iff (R,D) \text{ is achievable.}$$

- Some results required to prove the theorem. (def's)

(i)



Given $p_{X,\hat{X}}(x,\hat{x})$

$A^{(n)}_{d,\varepsilon} = \{ (\underline{x}^{(n)}, \hat{\underline{x}}^{(n)}) \in \mathcal{X}^n \times \hat{\mathcal{X}}^n :$

$\underline{x}^{(n)}$ is typical,

$\hat{\underline{x}}^{(n)}$ " " , and

$(\underline{x}^{(n)}, \hat{\underline{x}}^{(n)})$ is jointly typical,

$|d(\underline{x}^{(n)}, \hat{\underline{x}}^{(n)}) - E\{d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)})\}| \le \varepsilon \}$

(ii) Given $p_{X,\hat{X}}(x,\hat{x})$, if $(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)})$ is generated by using $p_{X,\hat{X}}(x,\hat{x})$ independently n times, then $Pr((\underline{X}^{(n)}, \hat{\underline{X}}^{(n)}) \in A^{(n)}_{d,\varepsilon}) \simeq 1$

(iii) " " , then $d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)}) \to E\{d(X,\hat{X})\}$

(iv) Need to link $p_X(x)p_{\hat{X}}(\hat{x})$ and $p_{X,\hat{X}}(x,\hat{x})$.

If $(\underline{x}^{(m)}, \hat{\underline{x}}^{(m)}) \in A^{(n)}_{d,\varepsilon}$, then

$$\cancel{\frac{p_X(x)p_{\hat{X}}(\hat{x})}{p_{X\hat{X}}(x,\hat{x})}} \qquad \frac{p(\underline{x}^{(m)})\,p(\hat{\underline{x}}^{(m)})}{p(\underline{x}^{(m)},\hat{\underline{x}}^{(m)})} \geq 2^{-n(I(X;\hat{X})+3\varepsilon)}$$

$$\Rightarrow \frac{p(\underline{x}^{(m)})\,p(\hat{\underline{x}}^{(m)})}{p(\underline{x}^{(m)})\,|}$$

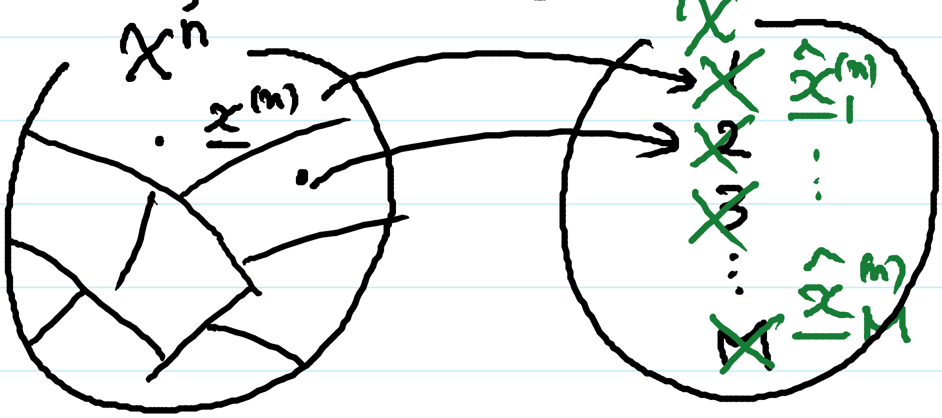
• Random Encoding

An (M, n) code is defined .....

rate distortion

observes the source output n times independently $\Rightarrow |\mathcal{X}|^n$

# of codewords



An (M, n) rate-distortion code $\mathcal{C}^{(n)}$ associated w/ distortion typical set ~~decoder~~ encoder.

$$\mathcal{C}^{(n)} = \{ \hat{\underline{x}}_1^{(n)}, \hat{\underline{x}}_2^{(n)}, \dots, \hat{\underline{x}}_M^{(n)} \}$$

We assume $p_{X,\hat{X}}(x,\hat{x})$ is given. $(\Rightarrow A_{d,\varepsilon}^{(n)})$

Given $\underline{x}^{(n)} \in \mathcal{X}^n$, $f_n(\underline{x}^{(n)}) = i$

if $(\underline{x}^{(n)}, \hat{\underline{x}}_i^{(n)}) \in A_{d,\varepsilon}^{(n)}$ &

$(\underline{x}^{(n)}, \hat{\underline{x}}_j^{(n)}) \notin A_{d,\varepsilon}^{(n)}$ $\quad j < i$,

$f_n(\underline{x}^{(n)}) = 1$ otherwise

- **Achievability**

If $R > R_I(D)$, then (R, D) is achievable.

(proof)

Suppose that a $(\lceil 2^{nR} \rceil, n)$ $\mathcal{C}^{(n)}$ & $\varepsilon > 0$, $P_{\hat{X}|X}(\hat{x}|x) = \min_{P_{\hat{X}|X} \text{ s.t. } E[d] \le D} I(X;\hat{X})$ are given.

$$E\{d(\underbrace{\underline{X}^{(n)}, \hat{\underline{X}}^{(n)}}_{}\)\} =$$

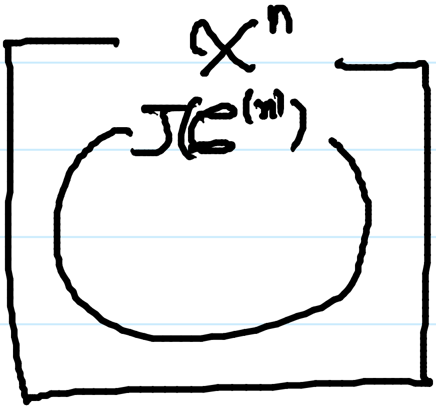
$\underline{X}^{(n)}$ → generated by $p_X(x)$ independently n times

$\hat{\underline{X}}^{(n)}$ → generated by distortion typical set encoder using $\mathcal{C}^{(n)}$ & $\underline{X}^{(n)}$

→ Not generated by using $p_{X,\hat{X}}(x,\hat{x})$ n times independently

- **Def**

$J(\mathcal{C}) \subset \mathcal{X}^n$



$$\{\underline{x}^{(n)} \in \mathcal{X}^n : (\underline{x}^{(n)}, \hat{\underline{x}}^{(n)}) \in A_{d,\varepsilon}^{(n)}\}$$

If $\underline{x}^{(n)} \in J(\mathcal{C}^{(n)})$, then $\underline{x}^{(n)}$ is a CAT-I seq.

$\underline{x}^{(n)}$ is a CAT-I seq.

Page 04

2013년 5월 29일 수요일

오후 9:03

- Return to $E\{d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)})\}$ $< D+\varepsilon$

$$E\{d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)})\} = E\{d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)}) \mid \underline{X}^{(n)} \in J(\mathcal{C}^{(n)})\} \times \Pr(\underline{X}^{(n)} \in J(\mathcal{C}^{(n)})) + E\{d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)}) \mid \underline{X}^{(n)} \notin J(\mathcal{C}^{(n)})\} \times \Pr(\underline{X}^{(n)} \notin J(\mathcal{C}^{(n)}))$$

< 1

$< d_{max}$

$\triangleq P_e(\mathcal{C}^{(n)})$

If $d(x, \hat{x}) \le d_{max}$, then

- $\overline{P_e}^{(n)}$

$$\overline{P_e}^{(n)} = \sum_{\mathcal{C}^{(n)}} \Pr(\mathcal{C}^{(n)}) P_e(\mathcal{C}^{(n)})$$

$$= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{\underline{x}^{(n)} \notin J(\mathcal{C})} \Pr(\underline{X}^{(n)} = \underline{x}^{(n)})$$

$$= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{\underline{x}^{(n)} \in \mathcal{X}^n} \Pr(\underline{X}^{(n)} = \underline{x}^{(n)}) 1_{J(\mathcal{C})^c}(\underline{x}^{(n)})$$

$$= \sum_{\mathcal{C}} \Pr(\mathcal{C}) \sum_{\underline{x}^{(n)} \in \mathcal{X}^n} \Pr(\underline{X}^{(n)} = \underline{x}^{(n)}) \prod_{i=1}^{\lceil 2^{nR} \rceil} 1_{A_{d,\varepsilon}^{(n)\,c}}(\underline{x}^{(n)}, \hat{\underline{x}}_i^{(n)})$$

$$= \sum_{\hat{\underline{x}}_1^{(n)} \in \hat{\mathcal{X}}^n} \sum_{\hat{\underline{x}}_2^{(n)} \in \hat{\mathcal{X}}^n} \cdots \sum_{\hat{\underline{x}}_{\lceil 2^{nR} \rceil}^{(n)} \in \hat{\mathcal{X}}^n} \prod_{i=1}^{\lceil 2^{nR} \rceil} p(\hat{\underline{x}}_i^{(n)}) \cdots$$

$$= \sum_{\underline{\hat{x}}_1^{(n)} \in \hat{\mathcal{X}}^n} \sum_{\underline{\hat{x}}_2^{(n)} \in \hat{\mathcal{X}}^n} \cdots \sum_{\underline{\hat{x}}_{2^{nR}}^{(n)} \in \hat{\mathcal{X}}^n} \prod_{i=1}^{n} p(\underline{x}_i^{(n)}) \cdots$$

Page 05

2013년 5월 29일 수요일

오후 9:03

$$\overline{P_e} = \sum_{\underline{x}^{(n)} \in \mathcal{X}^n} p(\underline{x}^{(n)}) \left(\sum_{\hat{\underline{x}}_1^{(n)} \in \hat{\mathcal{X}}^n} \cdots \sum_{\hat{\underline{x}}_{\lceil 2^{nR} \rceil}^{(n)} \in \hat{\mathcal{X}}^n} \prod_{i=1}^{\lceil 2^{nR} \rceil} p(\hat{\underline{x}}_i^{(n)}) \, 1_{A_{d,\varepsilon}^{(n)c}}(\underline{x}^{(n)}, \hat{\underline{x}}_i^{(n)}) \right)$$

$$\sum_{x \in A} \sum_{y \in A} f(x) f(y)$$

$$= \left(\sum_{x \in A} f(x)\right)\left(\sum_{y \in A} f(y)\right)$$

$$= \left(\sum_{x \in A} f(x)\right)^2$$

$$= \sum_{\underline{x}^{(n)} \in \mathcal{X}^n} p(\underline{x}^{(n)}) \left(\sum_{\hat{\underline{x}}^{(n)} \in \hat{\mathcal{X}}^n} p(\hat{\underline{x}}^{(n)}) \underbrace{1_{A_{d,\varepsilon}^{(n)c}}(\underline{x}^{(n)}, \hat{\underline{x}}^{(n)})}_{1 - 1_{A_{d,\varepsilon}^{(n)}}(\underline{x}^{(n)}, \hat{\underline{x}})} \right)^{\lceil 2^{nR} \rceil}$$

$$\underbrace{\left(1 - \sum_{\hat{\underline{x}}^{(n)} \in \hat{\mathcal{X}}^{(n)}} p(\hat{\underline{x}}^{(n)}) 1_{A_{d,\varepsilon}^{(n)}}(\ ,\ )\right)}_{\geq p(\hat{\underline{x}}^{(n)} | \underline{x}^{(n)}) 2^{-n(I(X;\hat{X})+3\varepsilon)}}^{\lceil 2^{nR} \rceil}$$

$$< \left(1 - \sum_{\hat{\underline{x}}^{(n)} \in \hat{\mathcal{X}}^{(n)}} p(\hat{\underline{x}}^{(n)} | \underline{x}^{(n)}) 1_{A_{d,\varepsilon}^{(n)}} + e \right)^{\lceil 2^{nR} \rceil}$$

$$< 1 - \sum_{\underline{x}^{(m)} \in \mathcal{X}^n} \sum_{\hat{\underline{x}}^{(m)} \in \hat{\mathcal{X}}^n} p(\underline{x}^{(m)}) p(\hat{\underline{x}}^{(m)} | \underline{x}^{(m)}) \, 1_{A_{d,\varepsilon}^{(m)}}(\underline{x}^{(m)}, \hat{\underline{x}}^{(m)})$$

$$+ e^{-\lceil 2^{nR} \rceil 2^{-n(I(X;\hat{X}) + 3\varepsilon)}}$$

$$< \frac{\delta}{2} + e^{-2^{n(R - R_I(D) - 3\varepsilon)}}$$

If $R > R_I(D)$ & ε is sufficiently small, then the RHS $\searrow 0$ as $n \to \infty$.

Thus, there exists $\mathcal{C}^{(m)*}$ s.t. $P_e(\mathcal{C}^{(m)*}) < \delta$, which implies

$$E\{d(X^{(m)}, \hat{X}^{(m)})\} \le D + \varepsilon + d_{max} \cdot \delta$$

Note that $\lim_{n \to \infty} a_n = a \le b$

$\Rightarrow$ $\forall \varepsilon > 0, \exists N(\varepsilon) \in \mathbb{N}$ s.t. "$n \ge N(\varepsilon) \Rightarrow$
$\underbrace{|a_n - a| < \varepsilon}_{a - \varepsilon < a_n < a + \varepsilon \le b + \varepsilon}$"

2013년 5월 29일 수요일

오후 9:03

- Converse.

$$\log(2^{nR}+1) \ge H(\underline{\hat{X}}^{(n)}) = H(\underline{\hat{X}}^{(n)}) - H(\underline{\hat{X}}^{(n)} | \underline{X}^{(n)})$$

$$= I(\underline{X}^{(n)}; \underline{\hat{X}}^{(n)})$$

$$= H(\underline{X}^{(n)}) - H(\underline{X}^{(n)} | \underline{\hat{X}}^{(n)})$$

$$= \sum_{i=1}^{n} H(X_i^{(n)}) - \sum_{i=1}^{n} H(X_i^{(n)} | \underline{\hat{X}}^{(n)}, X_1^{(n)}, \cdots, X_{i-1}^{(n)})$$

$$\ge \sum_{i=1}^{n} H(X_i^{(n)}) - \sum_{i=1}^{n} H(X_i^{(n)} | \hat{X}_i^{(n)})$$

$$= \sum_{i=1}^{n} I(X_i^{(n)}; \hat{X}_i^{(n)})$$

$$\ge \sum_{i=1}^{n} \frac{n}{n} R_I(E\{d(X_i^{(n)}, \hat{X}_i^{(n)})\})$$

A cvx ft on $S \subset \mathbb{R}^n$ is cont. on S.

$$\ge n R_I\left\{E\left(\sum_{i=1}^{n} \frac{1}{n} d(X_i^{(n)}, \hat{X}_i^{(n)})\right)\right\}$$

$$= n R_I\left(E\{d(\underline{X}^{(n)}, \underline{\hat{X}}^{(n)})\}\right)$$

$$\Rightarrow \frac{\log(2^{nR}+1)}{n} > R_I(E\{d(\underline{X}^{(n)}, \underline{\hat{X}}^{(n)})\})$$

Let $n \to \infty$, then

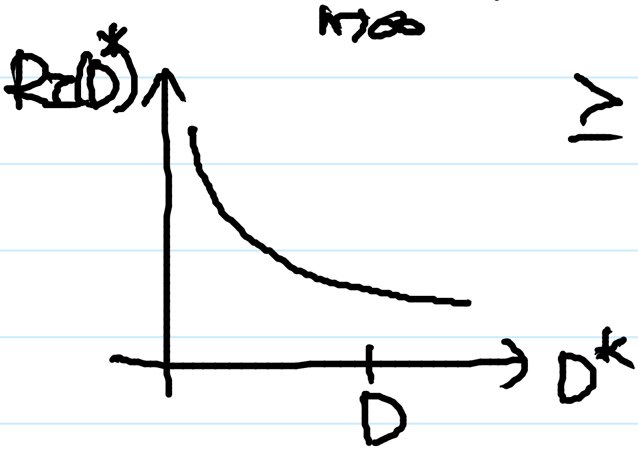
$$R \ge \lim_{n\to\infty} R_I(E\{d(\underline{X}^{(n)}, \underline{\hat{X}}^{(n)})\})$$

Page 08

2013년 5월 29일 수요일

오후 11:23

$$= R_I\left(\lim_{n\to\infty} E\{d(X^{(n)}, \hat{X}^{(n)})\}\right)$$

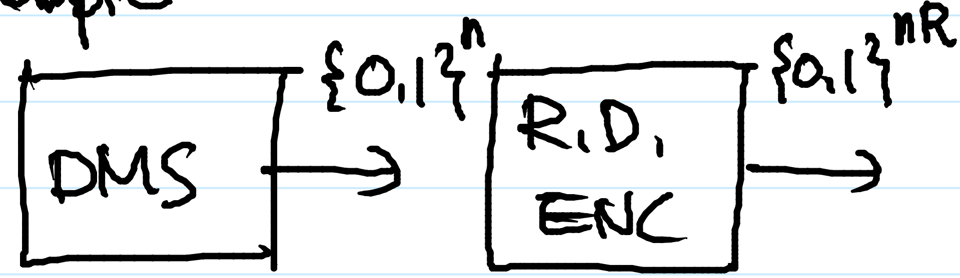


$$\ge R_I(D)$$

$$R \underset{(-)}{\ge} R_I(D).$$

- Rate distortion theorem.

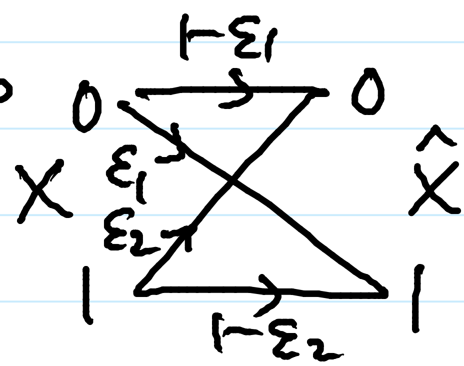
- Example



$p(0) = p(1) = 1/2$

$d(x, \hat{x}) = d_H(x, \hat{x})$

$E\{d(\underline{X}^{(n)}, \hat{\underline{X}}^{(n)})\} \le P_b$



To solve $\min_{P_{\hat{X}|X}} I(X; \hat{X})$

s.t. P_b

$$I(X;\hat{X})\Big|_{(\varepsilon_1,\varepsilon_2)} = I(X;\hat{X})\Big|_{(\varepsilon_2,\varepsilon_1)}$$

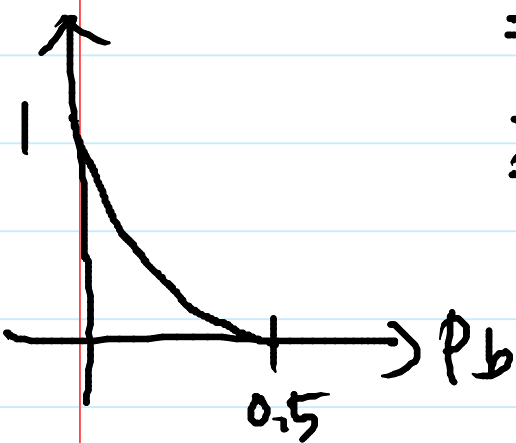
$$\geq I(X;\hat{X})\Big|_{\left(\frac{\varepsilon_1+\varepsilon_2}{2},\frac{\varepsilon_1+\varepsilon_2}{2}\right)}$$

Set $\varepsilon_1=\varepsilon_2 \triangleq P_b^{<0.5}$. Then,

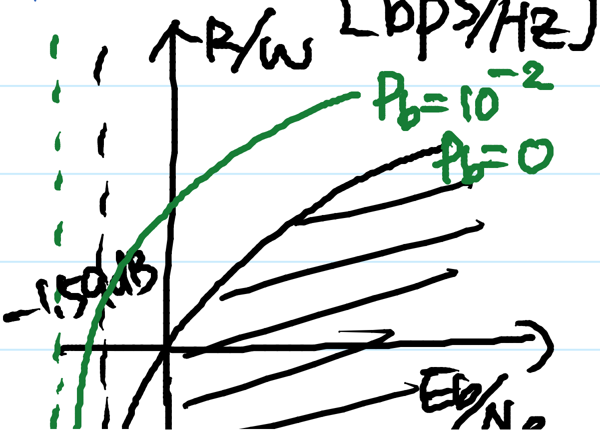
$$I(X;\hat{X}) = H(\hat{X}) - H(\hat{X}|X)$$

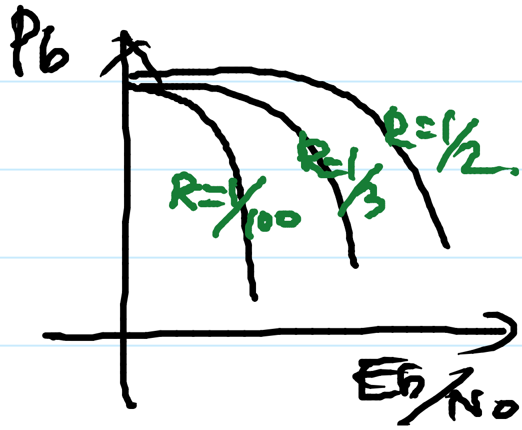
$$= 1 - h_b(\varepsilon)$$

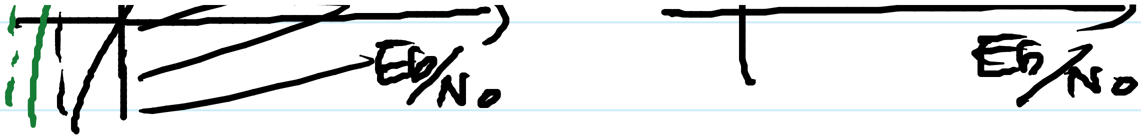
$$\geq 1 - h_b(P_b) \quad \left[\overset{\text{output}}{\text{bits}}/\overset{\text{input}}{\text{bits}}\right]$$



· Preview



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