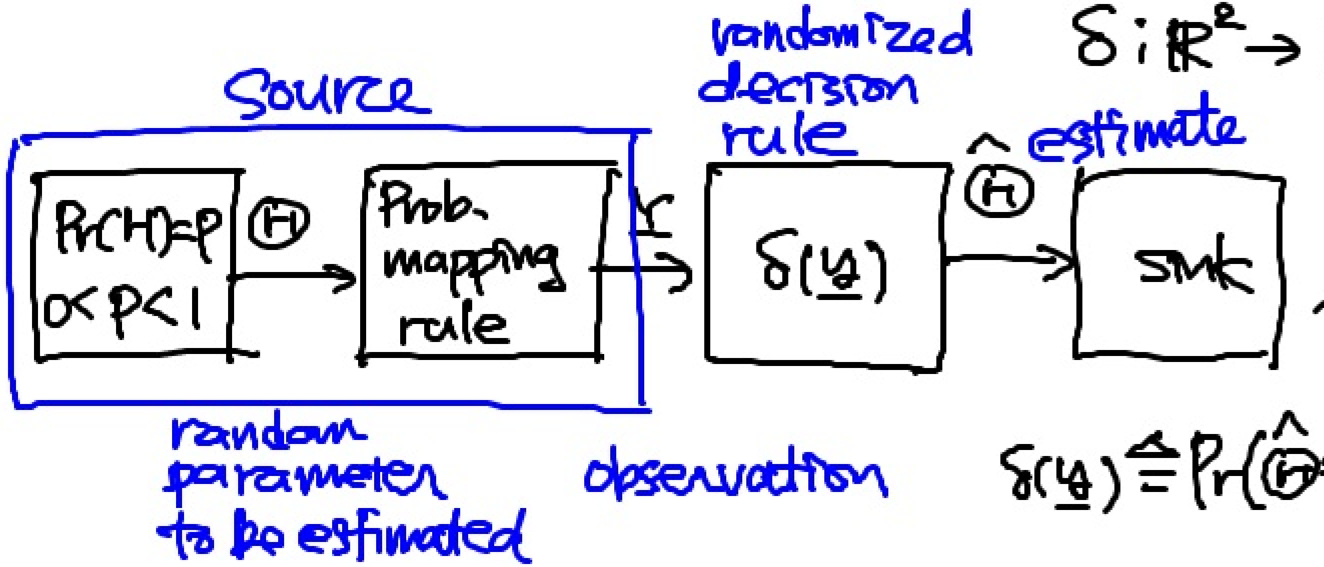
Lec. 03

• Review

Given



$\delta : \mathbb{R}^2 \to [0,1]$

$\delta(\underline{y}) \triangleq Pr(\hat{\Theta}=1 \mid \underline{Y}=\underline{y})$

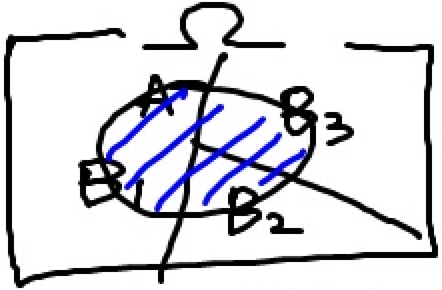
find $\quad \min_{\delta(\underline{y})} Pr(\Theta \neq \hat{\Theta})$

$\quad$ s.t. $0 \le \delta(\underline{y}) \le 1, \ \forall \underline{y} \in \mathbb{R}^2$

where $\quad f_{\underline{Y}}(\underline{y} \mid \Theta = i) = \dfrac{1}{(\sqrt{2\pi\sigma^2})^2} \exp\left(-\dfrac{\|\underline{y}-\underline{s}_i\|^2}{2\sigma^2}\right)$

$i=1,2.$

- Total Probability Theorem



$P(A\cap B_i)=P(A|B_i)P(B_i)$

$$P(A)=\sum_i P(A|B_i)P(B_i)$$

where $B_i\cap B_j=\phi,\ \forall i\neq j$ & $\bigcup_i B_i=\Omega$

($\bigcup_i B_i\supset A$ is also sufficient.)

p.3

· $\Pr(\Theta \neq \hat{\Theta}) =$ prob. of error

x^2-2x+3
$=(x-1)^2+2$

Representation (Rewriting).

(i) $\underline{\Pr(\Theta \neq \hat{\Theta} \mid \Theta=1)}\boxed{\Pr(\Theta=1)}$

$+\Pr(\Theta \neq \hat{\Theta} \mid \Theta=0)\boxed{\Pr(\Theta=0)}$

conditional prob. of error

Interpretation.

a priori probabilities

the prior ″

$\mathcal{D}$ $\{\Theta=1\}, \{\Theta=2\}$ ⋯ (i)

$\{\hat{\Theta}=1\}, \{\hat{\Theta}=2\}$ ⋯ (ii)

$\{\underline{Y}=\underline{y}\}$ $\underline{y}\in\mathbb{R}^2$ ⋯ (iii)

capable of exhaustive search.

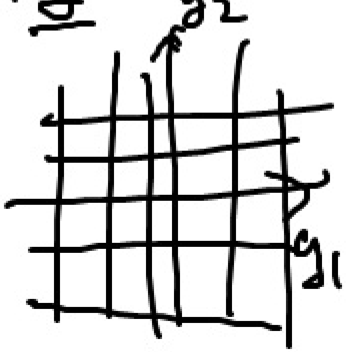
= intelligence.

$\boxed{5}+\boxed{5}+\boxed{5}+\boxed{5} = 4+4+4+4+4$

$= 4\times5$

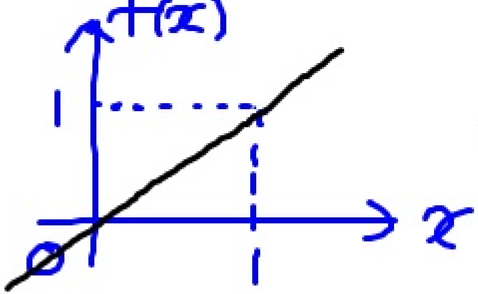
$= 5\times4$

Pf. $\cdot\ Pr(\Theta \neq \hat{\Theta}) = \int_{\underline{y}\in\mathbb{R}^2} Pr(\Theta\neq\hat{\Theta} \mid \underline{Y}=\underline{y})\, f_{\underline{Y}}(\underline{y})\, d\underline{y}$



$$f_{\underline{Y}}(\underline{y}) = f_{\underline{Y}|\Theta=1}(\underline{y})\, Pr(\Theta=1) + f_{\underline{Y}|\Theta=0}(\underline{y})\, Pr(\Theta=0)$$

Q. If $0 \le f(x) \le 1,\ \forall x \in [0, 1]$
then find $f(x)$ s.t. $\int_0^1 |f(x)-x|\,dx$
is maximized.



$$f(x) = \begin{cases} 1, & 0 \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1 \end{cases}$$

p.5

$$\min_{\delta(\underline{y})} \Pr(\Theta \neq \hat{\Theta}) \quad \text{s.t.} \quad 0 \le \delta(\underline{y}) \le 1, \ \forall \underline{y} \in \mathbb{R}^2$$

$$\equiv \left(\min_{\delta(\underline{y})} \Pr(\Theta \neq \hat{\Theta} \mid \underline{Y} = \underline{y}) \quad \text{s.t.} \quad 0 \le \delta(\underline{y}) \le 1 \right)_{\underline{y} \in \mathbb{R}^2}$$

\equiv

$$\Pr(\Theta \neq \hat{\Theta} \mid \underline{Y} = \underline{y}) = \Pr(\Theta \neq \hat{\Theta} \mid \overset{\{\hat{\Theta}=0\} \cap \{\underline{Y}=\underline{y}\}}{\hat{\Theta} = 0, \underline{Y} = \underline{y}}) \overbrace{\Pr(\hat{\Theta} = 0 \mid \underline{Y} = \underline{y})}^{1-\delta(\underline{y})}$$

$$+ \Pr(\Theta \neq \hat{\Theta} \mid \hat{\Theta} = 1, \underline{Y} = \underline{y}) \underbrace{\Pr(\hat{\Theta} = 1 \mid \underline{Y} = \underline{y})}_{\delta(\underline{y})}$$

and

cf)

$\Theta \to Y \to \hat{\Theta}$ forms a Markov chain.

the posterior prob.

$$\boxed{= \Pr(\Theta = 1 \mid \underline{Y} = \underline{y})\,(1 - \delta(\underline{y})) + \Pr(\Theta = 0 \mid \underline{Y} = \underline{y})\,\delta(\underline{y})}$$

a posterior prob.

$$P(A) = \sum_i P(A \cap B_i) = \sum_i P(A \mid B_i) P(B_i)$$

A posteriori prob

$$\Pr(\Theta = i \mid \underline{Y} = \underline{y}) = \frac{f_{\underline{Y}|\Theta=i}(\underline{y}) \Pr(\Theta = i)}{\sum_{j=0}^{1} f_{\underline{Y}|\Theta=j}(\underline{y}) \Pr(\Theta = j)}$$

The Bayes theorem (events version)

$$P(B_i \mid A) = \frac{P(B_i \cap A)}{P(A)} = \frac{P(A \mid B_i) P(B_i)}{\sum_j P(A \mid B_j) P(B_j)}$$

where $(B_i)_i$ forms a partition of Ω.

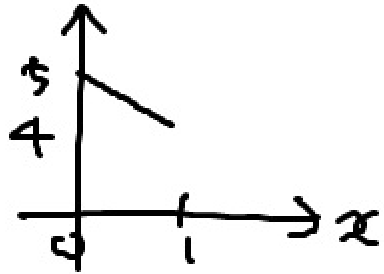
$$- \min_{\delta(\underline{y})} \Pr(\Theta \neq \hat{\Theta}) \equiv \min_{\delta(\underline{y})} \Pr(\Theta \neq \hat{\delta} \mid \underline{Y} = \underline{y})$$

$$\equiv \min \Pr(\Theta = 1 \mid \underline{Y} = \underline{y})(1 - \delta(\underline{y})) + \Pr(\Theta = 0 \mid \underline{Y} = \underline{y})\, \delta(\underline{y}))$$

p.7

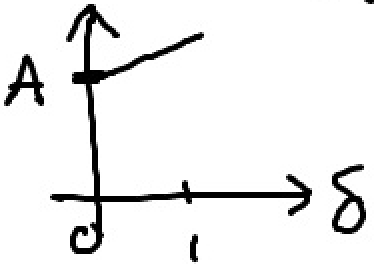
$$\min_{0<x<1} \; 5(1-x)+4x$$

$$5-5x+4x = 5-x$$



$$\min_{0<\delta<1} \; A(1-\delta)+B\delta$$

$$A - A\delta + B\delta = A-(A-B)\delta$$
$$= A+(B-A)\delta$$



If $B>A$, then $\delta_{opt}=0$

If $B=A$, then $0 \le \delta_{opt} \le 1$

If $B<A$, then $\delta_{opt}=1$

p.8 $\cancel{\min}$ $\Pr(\Theta=0 \mid \underline{Y}=\underline{y})\,\delta(\underline{y}) + \Pr(\Theta=1 \mid \underline{Y}=\underline{y})\,(1-\delta(\underline{y}))$

$$= \frac{f_{\underline{Y}|\Theta=0}(\underline{y})\Pr(\Theta=0)}{\cancel{f_{\underline{Y}}(\underline{y})}}\,\delta(\underline{y}) + \frac{f_{\underline{Y}|\Theta=1}(\underline{y})\Pr(\Theta=1)}{\cancel{f_{\underline{Y}}(\underline{y})}}\,(1-\delta(\underline{y}))$$

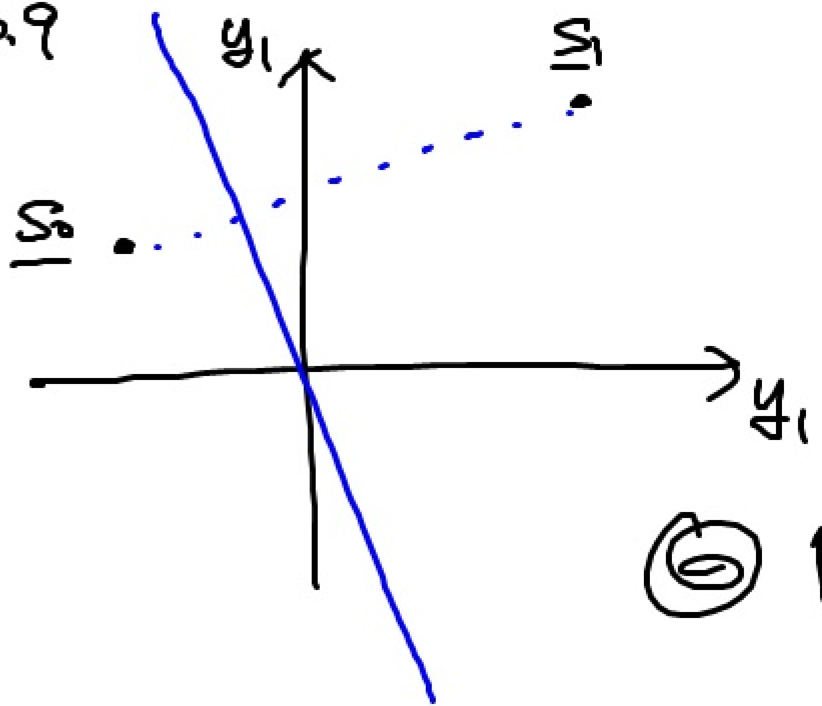
$$\equiv \delta(\underline{y}) = \begin{cases} 0, & \text{for } \dfrac{f_{\underline{Y}|\Theta=1}(\underline{y})}{f_{\underline{Y}|\Theta=0}(\underline{y})} \underset{(-)}{<} \dfrac{\Pr(\Theta=0)}{\Pr(\Theta=1)} \\[2ex] 1, & \text{for } \dfrac{f_{\underline{Y}|\Theta=1}(\underline{y})}{f_{\underline{Y}|\Theta=0}(\underline{y})} \underset{(-)}{>} \dfrac{\Pr(\Theta=0)}{\Pr(\Theta=1)} \end{cases}$$

likelihood ratio

LRT

test.

p.9



specialize ($\in$ simplify)

generalize

◎ parameter $\in \{T, H\}$

Binary Hypothesis testing

◎ M-ary Hypothesis Testing.