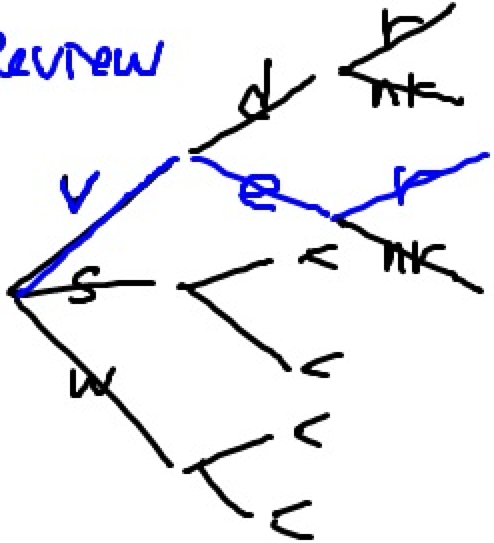
Lec. 14

Announcements
1. Make-ups on THIS Friday
2. Quiz #4 on NEXT Friday

Review



- $(\underline{Y}, \underline{\Theta})$ jointly distributed random vectors.

- $\underline{Y}=\underline{y} \rightarrow \boxed{\hat{\underline{\theta}}(\underline{y})} \rightarrow \hat{\underline{\Theta}} = \hat{\underline{\theta}}$ ← estimate

  estimator

  point estimator

- $\min E\left[C(\hat{\underline{\theta}}(\underline{Y}), \underline{\Theta})\right]$. $c(\hat{\underline{\theta}}, \underline{\theta})$

- $\hat{\underline{\theta}} - \underline{\theta}$: estimation error

MMAE
MAP
MMSE
LMMSE
AMMSE

p.2

Calculus of variations

변분법

分法

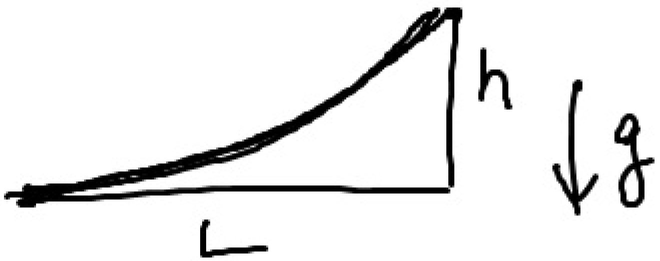
$$\min_{\hat{\underline{\theta}}(\underline{y})} \; E_{\underline{Y},\underline{\Theta}}\left[\|\hat{\underline{\theta}}(\underline{Y})-\underline{\Theta}\|^2\right]$$

variational problem E

$\boxed{\text{ratio}}$nal
irrational } numbers

$$= \min_{\hat{\underline{\theta}}(\underline{y})} \; E_{\underline{Y}}\left\{E_{\underline{\Theta}}\left[\|\hat{\underline{\theta}}(\underline{Y})-\underline{\Theta}\|^2 \,\middle|\, \underline{Y}\right]\right\}$$

$$g(\underline{y}) \triangleq E_{\underline{\Theta}}\left[\|\hat{\underline{\theta}}(\underline{y})-\underline{\Theta}\|^2 \,\middle|\, \underline{Y}=\underline{y}\right]$$



$$= \min_{\hat{\underline{\theta}}(\underline{y})} \; E_{\underline{Y}}\left[g(\underline{Y})\right]$$

$$\underline{Y} \rightarrow \boxed{g(\underline{y})} \rightarrow Z$$

$0 \le f(x) \le 1, \quad 0 \le x \le 1$

$$\max_{f(x)} \int_0^1 |f(x)-x| \, dx$$

p.3

$$\underset{\hat{\underline{\theta}}(\underline{y})}{\arg\min}\ E_{\underline{\Theta}}\left[\|\hat{\underline{\theta}}(\underline{y})-\underline{\Theta}\|^2 \,\middle|\, \underline{Y}=\underline{y}\right] = E\left[\underline{\Theta} \,\middle|\, \underline{Y}=\underline{y}\right]$$

MUSE estimator = the conditional mean estimator

1-D $E_{\underline{Y}}\left[Var(\Theta|\underline{Y})\right]$

N-D $tr\left\{E_{\underline{Y}}\left[Cov(\underline{\Theta}|\underline{Y})\right]\right\}$

$tr\{Cov(\underline{\Theta})\}$ NO! in general.

$\because E\{E[X|Y]\} = E\{X\}$

p.4

$$\|\underline{\hat{\theta}} - \underline{\theta}\|^2 \triangleq \sum_{n=1}^{N} 1 \cdot \left(\hat{\theta}_n - \theta_n\right)^2, \quad \text{where} \quad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{bmatrix}$$

ex) $$\underline{\theta} = \begin{array}{c} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ v_1 \\ v_2 \\ v_3 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{array} \begin{array}{l} \text{km} \\ \text{km} \\ \text{km} \\ \text{m/s} \\ \text{m/s} \\ \text{m/s} \\ \text{m/s}^2 \\ \text{m/s}^2 \\ \text{m/s}^2 \end{array}$$

Q. ⟨ generalize / specialize

$$\Rightarrow \sum_{n=1}^{N} C_n \left(\hat{\theta}_n - \theta\right)^2$$

$$\min_{\underline{\hat{\theta}}(\underline{y})} E\left\{ \sum_{n=1}^{N} c_n \left[\left(\hat{\theta}_n(\underline{Y}) - \Theta_n\right)^2 \right] \right\} ?$$

A. $E\left[\underline{\Theta} \mid \underline{Y} = \underline{y}\right]$

General square error objective function

$$c(\hat{\underline{\theta}}, \underline{\theta}) = (\hat{\underline{\theta}}-\underline{\theta})^T C (\hat{\underline{\theta}}-\underline{\theta}) \quad \text{where} \quad C > 0.$$

Problem Formulation

$$\hat{\underline{\theta}}(\underline{y}) = \arg\min_{\hat{\underline{\theta}}(\underline{y})} E\left[(\hat{\underline{\theta}}(\underline{Y})-\underline{\Theta})^T C (\hat{\underline{\theta}}(\underline{Y})-\underline{\Theta})\right]$$

Solving the problem

$$\hat{\underline{\theta}}(\underline{y}) \triangleq \arg\min_{\hat{\underline{\theta}}} E\left[(\hat{\underline{\theta}}-\underline{\Theta})^T C (\hat{\underline{\theta}}-\underline{\Theta}) \,\middle|\, \underline{Y}=\underline{y}\right], \quad \forall \underline{y}$$

By FONC, $\frac{\partial}{\partial \hat{\underline{\theta}}} E[\cdots] \Big|_{\hat{\underline{\theta}}=\hat{\underline{\theta}}(\underline{y})} = \underline{0}, \ \forall \underline{y}$

$$\frac{\partial}{\partial \underline{\hat{\theta}}}\left(\underline{\hat{\theta}}^T C \underline{\hat{\theta}}\right) = 2C\underline{\hat{\theta}}$$

$$\frac{\partial}{\partial \underline{\hat{\theta}}}\left(\underline{\mu}^T \underline{\hat{\theta}}\right) = \underline{\mu}$$

$$\frac{\partial}{\partial \underline{\hat{\theta}}}\left(\underline{\hat{\theta}}^T \underline{\mu}\right) = \underline{\mu}$$

$$\vdots$$

$$\frac{\partial}{\partial \underline{\hat{\theta}}} E\left[(\underline{\hat{\theta}}-\underline{\Theta})^T C (\underline{\hat{\theta}}-\underline{\Theta}) \mid \underline{Y}=\underline{y}\right]\Big|_{\underline{\hat{\theta}}=\underline{\hat{\theta}}(\underline{y})} = \underline{0}, \quad \forall \underline{y}$$

$$\equiv \quad 2E\left[C(\underline{\hat{\theta}}-\underline{\Theta}) \mid \underline{Y}=\underline{y}\right]\Big|_{\underline{\hat{\theta}}=\underline{\hat{\theta}}(\underline{y})} = \underline{0}, \quad \forall \underline{y}$$

$$\equiv \quad C\underline{\hat{\theta}}(\underline{y}) - C\,E\left[\underline{\Theta} \mid \underline{Y}=\underline{y}\right] = \underline{0}, \quad \forall \underline{y}$$

$\underline{\hat{\theta}} \in \mathbb{R}^M$.

Wirtinger Calculus if $\underline{\hat{\theta}} \in \mathbb{C}^M$.

proper-complex vector $\Leftrightarrow$ No problem.