Lec. 16

Review

when $\underline{\Theta}$ & $\underline{Y}$ are jointly Gaussian,

$$E[\underline{\Theta} \mid \underline{Y}=\underline{y}] = E[\underline{\Theta}] + \underbrace{\mathrm{Cov}(\underline{\Theta},\underline{Y})\,\mathrm{Cov}(\underline{Y})^{-1}}\,(\underline{y}-E[\underline{Y}])$$

$$\mathrm{Cov}[\underline{\Theta} \mid \underline{Y}=\underline{y}] = \mathrm{Cov}(\underline{\Theta}) - \underbrace{\mathrm{Cov}(\underline{\Theta},\underline{Y})\,\mathrm{Cov}(\underline{Y})^{-1}}\,\mathrm{Cov}(\underline{Y},\underline{\Theta})$$

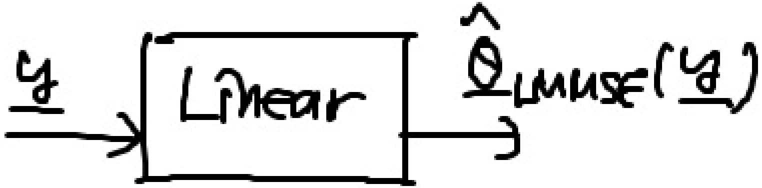
$$E\left[\left(\hat{\underline{\Theta}}_{MMSE}(\underline{Y})-\underline{\Theta}\right)^{T}\hat{\underline{\Theta}}_{MMSE}(\underline{Y})\right] = 0$$

Orthogonality Principle.

p.2

$$\hat{\underline{\theta}}_{LMMSE}(\underline{y}) = \arg\min_{\hat{\underline{\theta}}(\underline{y})} E\{\|\hat{\underline{\theta}}(\underline{Y})-\underline{\Theta}\|^2\}$$

s.t. $\hat{\underline{\theta}}(\underline{y}) = W\underline{y}$



$$tr(W R_{\underline{Y}\underline{Y}} W^T) = \sum_{i=1}^{M} \underline{w}_i^T R_{\underline{Y}\underline{Y}} \underline{w}_i \qquad -tr(W R_{\underline{Y}\underline{\Theta}}) = -\sum_{i=1}^{M} \underline{w}_i^T [R_{\underline{Y}\underline{\Theta}}]_{(:,i)}$$

where $W = \begin{bmatrix} \underline{w}_1^T \\ \underline{w}_2^T \\ \vdots \\ \underline{w}_M^T \end{bmatrix}$ $\quad \underline{y} \in \mathbb{R}^N$, $\underline{\theta} \in \mathbb{R}^M$ $\qquad -tr(R_{\underline{\Theta}\underline{Y}} W^T) =$ 〃

$$\cancel{2} R_{\underline{Y}\underline{Y}} \underline{w}_i = \cancel{2} [R_{\underline{Y}\underline{\Theta}}]_{(:,i)}$$

$$\Rightarrow \underline{w}_i = R_{\underline{Y}\underline{Y}}^{-1} [R_{\underline{Y}\underline{\Theta}}]_{(:,i)} \Rightarrow W^T = R_{\underline{Y}\underline{Y}}^{-1} R_{\underline{Y}\underline{\Theta}} \Rightarrow W = R_{\underline{\Theta}\underline{Y}} R_{\underline{Y}\underline{Y}}^{-1}$$

p.3 Def. Let $\tilde{W}$ be defined as an $M \times N$ matrix s.t.

existence

uniqueness

$$E\{(\underline{\Theta} - \tilde{W}\underline{Y})\underline{Y}^T\} = \underline{O}$$

Theorem

$$\underbrace{E\{\|\tilde{W}\underline{Y} - \underline{\Theta}\|^2\}}_{\substack{\text{MSE when} \\ \tilde{W}\underline{y} \text{ is used}}} \le \underbrace{E\{\|W\underline{Y} - \underline{\Theta}\|^2\}}_{\substack{\text{MSE when} \\ W\underline{y} \text{ is used}}}, \forall W$$

$\Rightarrow$ $W_{LMMSE} = \tilde{W}$.

Lemma

$$E\{\underline{\Theta}\underline{Y}^T\} = \tilde{W} E\{\underline{Y}\underline{Y}^T\}$$

$$\Rightarrow R_{\underline{\Theta}\underline{Y}} = \tilde{W} R_{\underline{Y}\underline{Y}} \Rightarrow \tilde{W} = R_{\underline{\Theta}\underline{Y}} R_{\underline{Y}\underline{Y}}^{-1}$$

p.4 $\hat{\underline{\theta}}_{AMMSE}(\underline{y}) = \arg\min_{W,\underline{b}} E\{\|W\underline{Y}+\underline{b}-\underline{\Theta}\|^2\}$

$$= \arg\min_{W} \underbrace{\left(\min_{\underline{b}} E\{\|W\underline{Y}+\underline{b}-\underline{\Theta}\|^2\}\right)}$$

$$\underline{b}_{opt} = E[\underline{\Theta} - W\underline{Y}]$$

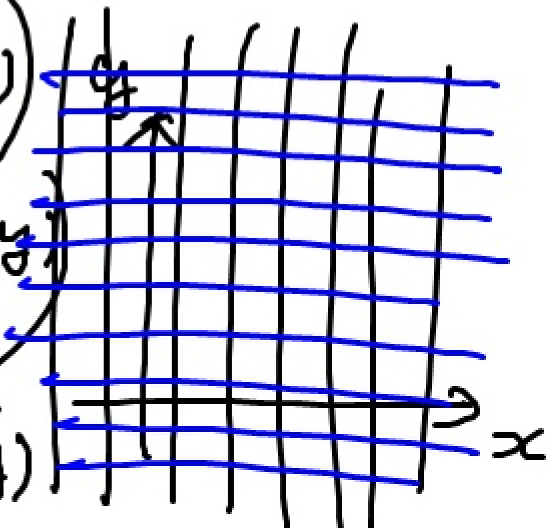
$$= E[\underline{\Theta}] - W E[\underline{Y}]$$

$$\arg\min_{W} E\{\|W\underline{Y} + E[\underline{\Theta}] - W E[\underline{Y}] - \underline{\Theta}\|^2\}$$

$$\min_{(x,y)} f(x,y)$$

$$= \min_{x} \underbrace{\left(\min_{y} f(x,y)\right)}_{g(x)}$$

$$= \min_{y} \underbrace{\left(\min_{x} f(x,y)\right)}_{h(y)}$$



$$\Rightarrow \arg\min_{W} E\left[\left\| W(\underbrace{\underline{Y}-E\{\underline{Y}\}}_{\triangleq \underline{Z}}) - (\underbrace{\underline{\Theta}-E\{\underline{\Theta}\}}_{\triangleq \underline{\Phi}}) \right\|^2 \right]$$

$$\Rightarrow \quad W = R_{\underline{\Phi}\underline{Z}} R_{\underline{Z}\underline{Z}}^{-1}$$

$$= E[\underline{\Phi}\,\underline{Z}^T]\left(E[\underline{Z}\,\underline{Z}^T]\right)^{-1}$$

$$= E\{(\underline{\Theta}-E[\underline{\Theta}])(\underline{Y}-E[\underline{Y}])^T\}\left(\quad\right)^{-1}$$

$$= Cov(\underline{\Theta},\underline{Y})\,Cov(\underline{Y})^{-1}$$

$$W\underline{y}+\underline{b} = E\{\underline{\Theta}\}\cancel{-WE\{\underline{Y}\}} + Cov(\underline{\Theta},\underline{Y})Cov(\underline{Y})^{-1}(\underline{y}-E[\underline{Y}])$$

p.6 Theorem.

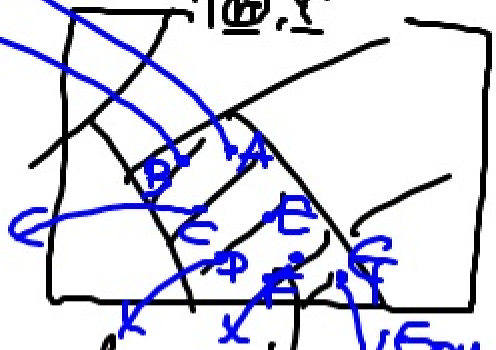
Given $\underline{\Theta}$ and $\underline{Y}$ with $E\{\underline{\Theta}\}$, $E\{\underline{Y}\}$, $Cov(\underline{\Theta})$, $Cov(\underline{Y})$ and $Cov(\underline{\Theta}, \underline{Y})$

the MSE performance of the MMSE estimator becomes the worst when $\underline{\Theta}$ & $\underline{Y}$ are jointly Gaussian.

$MMSE_A \le AMMSE_A$

$MMSE_B \le AMMSE_B$

$f_{\underline{\Theta},\underline{Y}}$



Gaussian

same 2nd order property

$MMSE_G \le AMMSE_G$

$\Rightarrow MMSE_j \le AMMSE = MMSE_G$