Lec. 20

Announcement

Midterm Exam #2 on 11/15.

Review

- A family of distribution functions

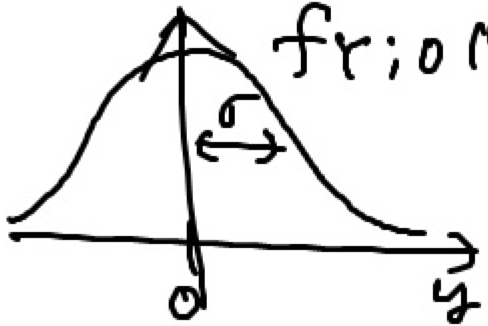
$$\{ f_{\underline{Y};\underline{\theta}}(\underline{y}) ; \underline{\theta} \in A \}$$

- a statistic $\quad \underline{Y} \longrightarrow \boxed{\underline{T}(\underline{y})} \longrightarrow \underline{T}(\underline{Y})$

- a sufficient statistic : $\underline{T}(\underline{Y})$ is a s.s. $\Leftrightarrow$ 

p.2 ex1

$f_{Y;0}(y)$

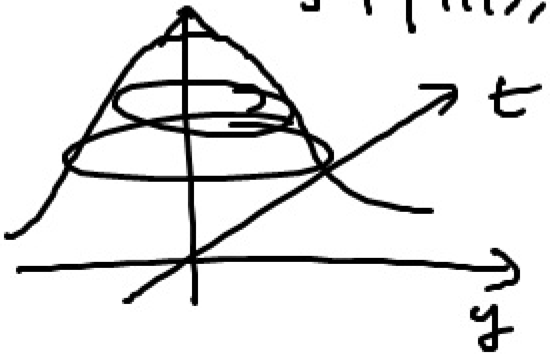


$f_{Y;1}(y)$

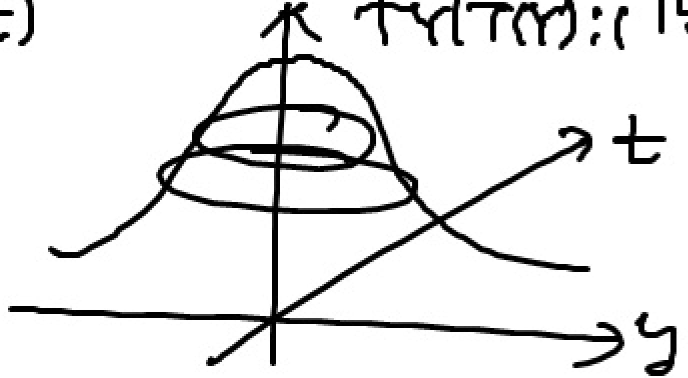
"We don't need to observe Y."

ex1

$f_{Y|T(Y);0}(y|t)$



$f_{Y|T(Y);1}(y|t)$



"We don't need to observe Y given $\{T(Y)=t\}$."

"$f_{\underline{Y}|\underline{T}(\underline{Y});\underline{\theta}}(\underline{y}|\underline{t})$ not a function of $\underline{\theta}$, $\forall \underline{y}, \forall \underline{t}$"

p3 " a sufficient statistic for $\underline{\theta}$ "

for a family of distribution functions $\{f_{\underline{Y};\underline{\theta}}(\underline{y}); \underline{\theta} \in A\}$ parameterized by $\underline{\theta}$.

The trouble w/ definition:

$f_{\underline{Y}|\underline{T}(\underline{Y});\underline{\theta}}(\underline{y}|\underline{t})$ not a function of $\underline{\theta}$.

1) $\underline{T}(\underline{Y})$ must be given.
2) Need to compute $f_{\underline{Y}|\underline{T}(\underline{Y});\underline{\theta}}(\underline{y}|\underline{t})$
3) Need to check whether it is a ft of $\underline{\theta}$ or not

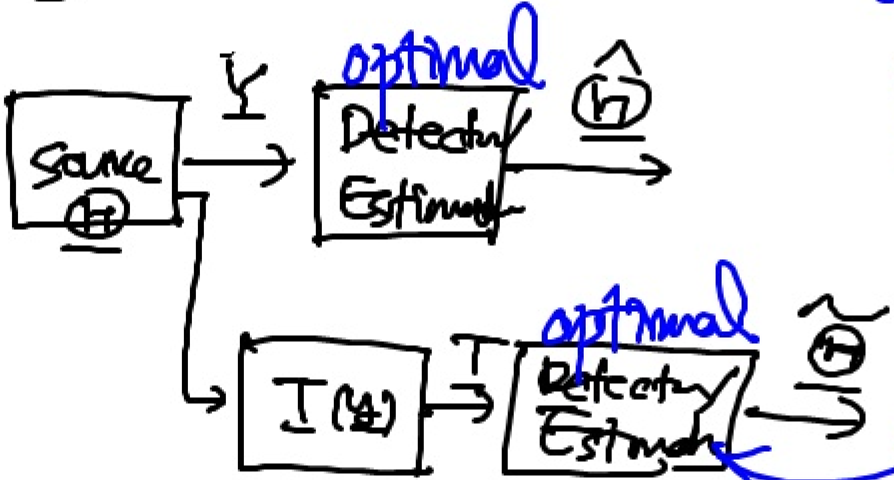
pf

$\underline{\theta}$ $\quad f_{\underline{Y}|\underline{T}(\underline{Y});\underline{\theta}}(\underline{y}|\underline{t})$ ...

$\underline{\Theta}$ $\quad \underbrace{f_{\underline{Y}|(\underline{T}(\underline{Y}),\underline{\Theta})}(\underline{y}|\underline{t},\underline{\theta})}$ $\qquad f_{\underline{Y}|\underline{\Theta}}(\underline{y}|\underline{\theta})$

Theorem.

If $\underline{\Theta}$ is random then $f_{\underline{\Theta}|\underline{Y}}(\underline{\theta}|\underline{y})$ is a function of $\underline{y}$ & $\underline{\theta}$ only through $\underline{T}(\underline{y})$.



$\{f_{\underline{T}|\underline{\Theta}}(\underline{t}|\underline{\theta})\}$

p5 Given $p_{\underline{Y}|\underline{T}(\underline{Y}),\underline{\Theta}}(\underline{y}|\underline{t},\underline{\theta})$ not a function of $\underline{\theta}$,

show, $p_{\underline{\Theta}|\underline{Y}}(\underline{\theta}|\underline{y})$ is a function of $\underline{\theta}$ & $\underline{T}(\underline{y})$.

sol)

$$p_{\underline{Y}|\underline{T}(\underline{Y}),\underline{\Theta}}(\underline{y}|\underline{t},\underline{\theta}) = Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{Y})=\underline{t}, \underline{\Theta}=\underline{\theta})$$

$$= Pr(\underline{\Theta}=\underline{\theta} \mid \{\boxed{\underline{Y}=\underline{y}}\} \cancel{\{\underline{T}(\underline{Y})=\underline{t}\}}) \cdot \frac{Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{Y})=\underline{t})}{Pr(\underline{\Theta}=\underline{\theta} \mid \underline{T}(\underline{Y})=\underline{t})}$$

$= \text{not a function of } \underline{\theta}$.

$\Longleftrightarrow$

p.6

We start from "$P_{\underline{Y}|\underline{T}(\underline{Y}),\underline{\Theta}}(\underline{y}|\underline{t},\underline{\theta})$ is not a function of $\underline{\theta}$".

$$\Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{Y})=\underline{t}, \underline{\Theta}=\underline{\theta}) = \begin{cases} 0, & \text{if } \underline{t} \neq \underline{T}(\underline{y}) \\ \Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{Y})=\underline{T}(\underline{y}), \underline{\Theta}=\underline{\theta}), & \text{otherwise } (\underline{t}=\underline{T}(\underline{y})) \end{cases}$$

$$\Pr(\underline{Y}=\underline{y} \mid \underline{T}(\underline{Y})=\underline{T}(\underline{y}), \underline{\Theta}=\underline{\theta}) = \cdots$$

P7

Example 2. $\underline{Y} = A\underline{1} + \underline{N}$

" $f_{A|\underline{Y}}(a|\underline{y})$ is a function of a & $\underline{1}^T\underline{y}$."

$$\mu \triangleq E[A|\underline{Y}=\underline{y}] = E[A] + \mathrm{Cov}(A,\underline{Y})\,\mathrm{Cov}(\underline{Y})^{-1}(\underline{y}-E[\underline{Y}])$$

$$= 0 + \sigma_a^2(\sigma_a^2\underline{1}\underline{1}^T + \sigma_n^2 I)^{-1}\underline{y}$$

$$\sigma^2 \triangleq \mathrm{Var}[A|\underline{Y}=\underline{y}] = \mathrm{Var}[A] - \mathrm{Cov}(A,\underline{Y})\,\mathrm{Cov}(\underline{Y})^{-1}\mathrm{Cov}(\underline{Y},A)$$

$$= \sigma_a^2 - \sigma_a^2(\sigma_a^2\underline{1}\underline{1}^T + \sigma_n^2 I)^{-1}\sigma_a^2.$$

$$\rightarrow \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(a-\mu)^2}{2\sigma^2}\right)$$

p.8 Matrix Inversion Lemma.

$(A+BC)^{-1} = \cdots$ in terms of A^{-1} ...

$(A+\underline{p}\underline{b}^T)^{-1} = \cdots$ " " " A^{-1} ...

$(A+\underline{p}\underline{p}^T)^{-1} = \cdots$ " " " A^{-1} ..

The Sherman-Morrison Formula.

$$A^{-1} - \frac{A^{-1}\underline{p}\,\underline{b}^T A^{-1}}{1+\underline{b}^T A^{-1}\underline{p}}$$

P9

$$E[A \mid \underline{Y} = \underline{y}] = \underbrace{\sigma_a^2}\,\underline{1}^T \Big(\underbrace{\sigma_a^2 \underline{1}}_{P}\,\underbrace{\underline{1}^T}_{R^T} + \sigma_n^2 I \Big)^{-1} \underline{y}$$

$$= \sigma_a^2 \underline{1}^T \left(\frac{I}{\sigma_n^2} - \frac{\frac{I}{\sigma_n^2}\, \sigma_a^2 \underline{1}\, \underline{1}^T \frac{I}{\sigma_n^2}}{1 + \underline{1}^T \frac{I}{\sigma_n^2}\, \sigma_a^2 \underline{1}} \right) \underline{y}$$

$$= \sigma_a^2 \underline{1}^T \left(\frac{\underline{y}}{\sigma_n^2} - \frac{\frac{\sigma_a^2}{\sigma_n^2} \underline{1}\; \frac{\underline{1}^T \underline{y}}{\sigma_n^2}}{1 + \frac{N}{\sigma_n^2} \sigma_a^2} \right)$$

$$= \frac{\sigma_a^2}{\sigma_n^2}\, \underline{1}^T \underline{y} - \frac{\frac{\sigma_a^4}{\sigma_n^2} N\; \frac{\underline{1}^T \underline{y}}{\sigma_n^2}}{1 + \frac{N}{\sigma_n^2} \sigma_a^2} = \cdots =$$

$$= \left(\frac{\sigma_a^2}{\sigma_n^2} - \frac{\frac{\sigma_a^4}{\sigma_n^4} N}{1 + \frac{\sigma_a^2}{\sigma_n^2} N} \right) \underline{1}^T \underline{y}$$

p10

S.S. $\underline{\theta}$ / $\underline{\Theta}$ — definition

$f_{\underline{\Theta}|\underline{Y}}(\underline{\theta}|\underline{y})$ is a function only of $\underline{\theta}$ & $\underline{T}(\underline{y})$.

☹ Must have a candidate $\underline{T}(\underline{Y})$.

$\updownarrow$

Neyman-Fisher Factorization theorem.

$\underline{T}(\underline{Y})$ is sufficient $\Leftrightarrow$ $f_{\underline{Y};\underline{\theta}}(\underline{y}) = \cdots$ $\forall \underline{y}$, $\forall \underline{\theta}$.

p.11

$$f_{\underline{Y};\underline{\theta}}(\underline{y}) = \underbrace{\text{a ft only of } \underline{y}}\ \times\ \underbrace{\text{a ft of } \underline{T}(\underline{y})\ \&\ \underline{\theta}}\quad \forall \underline{y}, \forall \underline{\theta}$$

a function of $\underline{T}(\underline{y})$.

$\Leftrightarrow$ $\underline{T}(\underline{Y})$ is sufficient.

a ft of $\underline{\theta}$.

$\Leftrightarrow$ $\exists$ $\underline{T}(\underline{y})$, $\underbrace{h(\underline{y})}_{\text{co-factor.}}$,

$g(\underline{t}, \underline{\theta})$

s.t. $f_{\underline{Y};\underline{\theta}}(\underline{y}) = h(\underline{y})\, g(\underline{T}(\underline{y}), \underline{\theta})$;

$\forall \underline{y}, \forall \underline{\theta}$.

p.12.

Example 1. $\underline{Y} = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_N \end{bmatrix}$ Y_i's i.i.d.

$Pr(Y=1) = \theta = 1 - Pr(Y=0)$

$$p_{\underline{Y};\theta}(\underline{y}) = Pr(\underline{Y} = \underline{y}; \theta) = \prod_{i=1}^{N} Pr(Y_i = y_i; \theta)$$

$$= \prod_{i=1}^{N} \theta^{y_i} (1-\theta)^{1-y_i}$$

$$= \theta^{\Sigma y_i} (1-\theta)^{N - \Sigma y_i}$$

To apply the N-F factorization theorem, we search for $\underset{\Sigma y_i = \underline{1}^T \underline{y}}{T(\underline{y})}$, $h(\underline{y}) = 1$, $\underset{\theta^t (1-\theta)^{N-t}}{g(t, \underline{\theta})}$.